Groupoids in Operator Algebra and Abstract Algebra: Part 1

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Becky Armstrong

VICTORIA UNIVERSITY OF WELLINGTON
TE HERENGA WAKA

Groupoids and Puzzles

Rubik's Cubes

Goal: Obtain a certain arrangement by performing a correct sequence of transformations.

Question: How do we view the transformations of a Rubik's Cube as a group?

Let F, B, L, R, U, and D denote 90° clockwise rotations of the 6 faces of the cube.

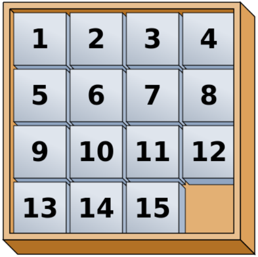
These rotations – modulo some relations – generate the **Rubik's Cube group**.

This group has approximately 4.33×10^{19} elements!

The fastest ever solve of a scrambled Rubik's cube is 3.05 seconds.

The Fifteen Puzzle

Goal: Obtain a certain arrangement by performing a correct sequence of transformations.



Question: Do the transformations of the Fifteen Puzzle also have a group-like structure?

Transformations of the Fifteen Puzzle are compositions of movements of the blank square in any of the directions Up, Down, Left, or Right, from any of the 16 different spaces.

There are approximately 1.67×10^{14} different transformations of the Fifteen Puzzle!

Problem: Unlike the Rubik's Cube, the possible transformations depend on the current state. The transformations do not form a group because we can only compose two transformations if the ending position of the first is the starting position of the second.

What is a groupoid?

A **groupoid** $\mathcal{G}$ is a small category in which every morphism γ has a unique inverse γ^{-1}.

Each $\gamma \in \mathcal{G}$ has a **range** $\mathbf{r}(\gamma) = \gamma\gamma^{-1}$ and a **source** $\mathbf{s}(\gamma) = \gamma^{-1}\gamma$.

Composition $(\alpha, \beta) \mapsto \alpha\beta$ is only defined on the set of **composable pairs**

$$\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid \mathbf{s}(\alpha) = \mathbf{r}(\beta)\} \subseteq \mathcal{G} \times \mathcal{G}.$$

The elements of the **unit space** $\mathcal{G}^{(0)} = \mathbf{r}(\mathcal{G}) = \mathbf{s}(\mathcal{G})$ behave like identities wherever composition is defined.

Examples

- If $\mathcal{G}^{(0)}$ is a singleton set, then $\mathcal{G}^{(2)} = \mathcal{G} \times \mathcal{G}$, and thus $\mathcal{G}$ is a group.
- If $G_1, \ldots, G_n$ are groups, then $\mathcal{G} := \bigsqcup_{i=1}^{n} G_i$ is a groupoid with $\mathcal{G}^{(0)} = \{\mathrm{id}_{G_i}\}_{i=1}^{n}$.

Groups and groupoids

A groupoid is a generalisation of a group with **multiple identities** ("dots") and **partially defined multiplication** ("concatenating arrows").

Groupoid example: equivalence relations

Let X be a set. An equivalence relation $R \subseteq X \times X$ is a groupoid with

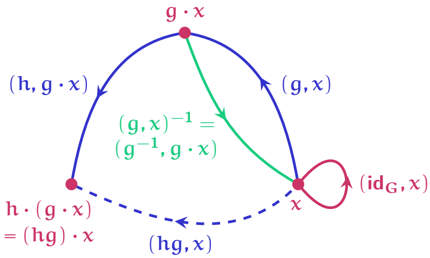
- inverses: $(x, y)^{-1} = (y, x) \in R$ for all $(x, y) \in R$;
- composable pairs: $R^{(2)} = \{\big((x, y), (w, z)\big) \in R \times R \mid y = w\}$;
- composition: $(x, y)(y, z) = (x, z) \in R$;
- range and source maps: $\mathbf{r}(x, y) = (x, x)$ and $\mathbf{s}(x, y) = (y, y)$;
- unit space: $R^{(0)} = \mathbf{r}(R) = \{(x, x) \mid x \in X\} \cong X$.

Groupoid example: group actions

Suppose that G is a group acting on some set X.

Then $G \ltimes X := G \times X$ is a groupoid with unit space $\{\mathrm{id}_G\} \times X \cong X$.

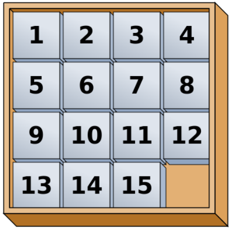
Each $(g, x) \in G \ltimes X$ is a morphism with source x and range $g \cdot x$.



Composition is given by $(h, g \cdot x)(g, x) = (hg, x)$.

Each $(g, x) \in G \ltimes X$ has a unique inverse $(g, x)^{-1} = (g^{-1}, g \cdot x) \in G \ltimes X$.

A groupoid model for the Fifteen Puzzle



The **groupoid** associated to the Fifteen Puzzle consists of all possible transformations from one state to another, obtained by **composing** Up, Down, Left, Right, and trivial movements of the blank square.

Each transformation has a unique **inverse**. The **unit space** is the set of the 16 possible positions of the blank square.

Two transformations are **equivalent** if they have the same starting and ending positions for the blank square and they affect all of the numbered pieces in the same way.

Two transformations can only be **composed** if the ending position of the first is the starting position of the second.

Isotropy: groups within a groupoid

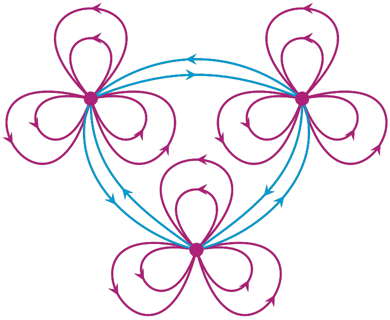
Let $\mathcal{G}$ be a groupoid. For each $x \in \mathcal{G}^{(0)}$, the set

$$\mathrm{Iso}(\mathcal{G})_x \coloneqq \{\gamma \in \mathcal{G} \mid \mathbf{r}(\gamma) = \mathbf{s}(\gamma) = x\}$$

is called the **isotropy group** of $\mathcal{G}$ at x.

The **isotropy groupoid** of $\mathcal{G}$ is the subgroupoid

$$\mathrm{Iso}(\mathcal{G}) \coloneqq \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathrm{Iso}(\mathcal{G})_x \subseteq \mathcal{G}.$$



Examples

- If $\mathcal{G}$ is a group, then $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}$.
- If $R \subseteq X \times X$ is an equivalence relation, then $\mathrm{Iso}(R) = \{(x, x) \mid x \in X\} = R^{(0)} \cong X$.
- If $G \curvearrowright X$, then $\mathrm{Iso}(G \ltimes X) = \{(g, x) \in G \times X \mid g \cdot x = x\} \cong \bigsqcup_{x \in X} \mathrm{Stab}_G(x)$.

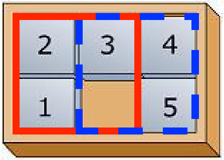
A solution strategy for the Fifteen Puzzle

The basic idea is to take advantage of **isotropy groups** for several simpler sub-puzzles.

Each isotropy group of the **Three Puzzle** is the cyclic group of order 3.

To solve the Three Puzzle, rotate the blank square clockwise (or anticlockwise) until the correct arrangement is obtained.

The **Five Puzzle** can be viewed as two overlapping Three Puzzles.

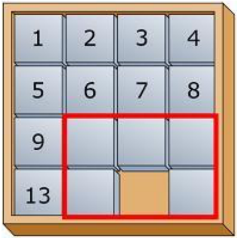


To solve the Five Puzzle, solve the left and right copies of the Three Puzzle using transformations that keep the blank square in the bottom middle position.

An algorithm for solving the Fifteen Puzzle

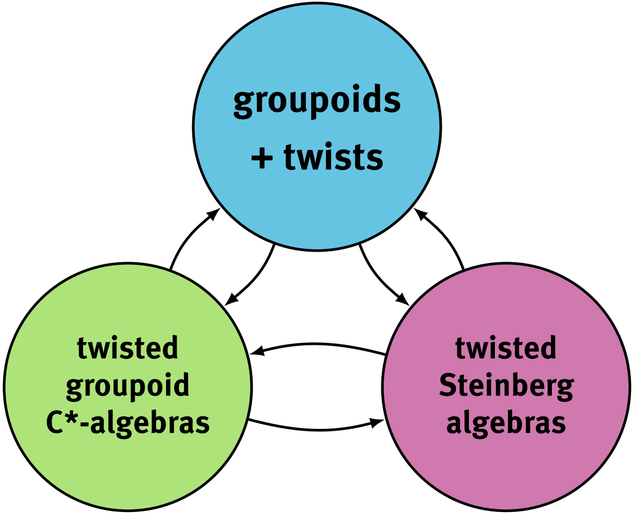
To solve the Fifteen Puzzle:

1. Move the **1** and **2** tiles into their correct positions.
2. Play the Three Puzzle multiple times to position the **3, 4, 5, 6, 7,** and **8** tiles. (The tricky part here is making sure that you form solvable Three Puzzles.)
3. Play the Five Puzzle in the lower left region to position the **9** and **13** tiles.
4. Play the Five Puzzle in the lower right region to position the remaining tiles.



Groupoids in Operator Algebra and Abstract Algebra

My research: twisted groupoid algebras



$*$-algebras

A **$*$-algebra** is a vector space over $\mathbb{C}$ (or a module over a ring with an involution) with an associative multiplication and an involution $*$ that is conjugate-linear and antimultiplicative.

Examples

- $\mathbb{C}$ with complex conjugation as the involution.
- $M_n(\mathbb{C})$ with the adjoint (conjugate transpose) as the involution.
- $\mathbb{C}[x] = \text{span}_{\mathbb{C}}\{x^n \mid n \in \mathbb{N}\}$ with coordinatewise complex conjugation as the involution.
- If R is a commutative unital ring with an involution $\dagger$, then $R[x] = \text{span}_R\{x^n \mid n \in \mathbb{N}\}$ is a $*$-algebra over R with involution given by $\left(\sum_{n=0}^{N} r_n x^n\right)^* = \sum_{n=0}^{N} r_n^\dagger x^n$.

C*-algebras

A **C*-algebra** is a $*$-algebra A over $\mathbb{C}$ that is complete with respect to a submultiplicative norm (i.e. $\|ab\| \leqslant \|a\|\|b\|$) that satisfies the **C*-identity:** $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Examples

Algebra	Involution	Norm
$\mathbb{C}$	complex conjugation	magnitude
$M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$	conjugate transpose	operator norm
$C(X)$ for a compact Hausdorff space X, or $C_0(Y)$ for a locally compact Hausdorff space Y	pointwise complex conjugation	uniform norm
$B(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$	adjoint	operator norm

The Gelfand–Naimark theorem

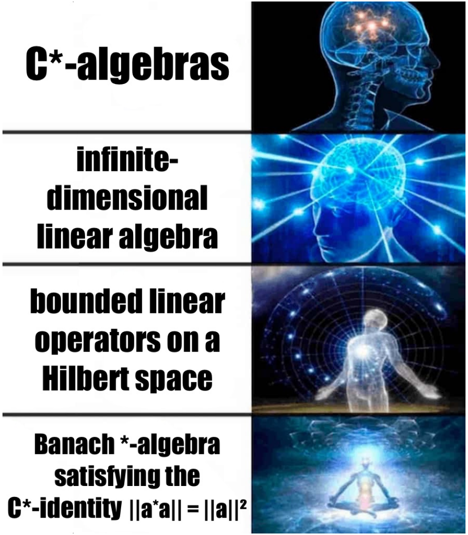
Theorem (Gelfand–Naimark 1943, Gelfand–Naimark–Segal 1947)

(a) *Every C*-algebra is isomorphic to a norm-closed *-subalgebra of the bounded linear operators on a Hilbert space. (We can explicitly construct the Hilbert space and the isomorphism through a process called the GNS construction.)*

(b) *Every commutative C*-algebra is isomorphic to* $C_0(X)$ *for some locally compact Hausdorff space* X.

Informally: A C*-algebra can be thought of as a collection of infinite-dimensional matrices.

C*-algebras and the Gelfand–Naimark theorem



Group $*$-algebras

Let G be a discrete group. The collection $\mathbb{C}[G]$ of finitely supported $\mathbb{C}$-valued functions on G is a vector space under pointwise operations, with a basis $\{\delta_x \mid x \in G\}$ of point mass functions

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

For all $x, y \in G$, define

$$\delta_x * \delta_y := \delta_{xy} \qquad \text{and} \qquad \delta_x^* := \delta_{x^{-1}}.$$

Extending this convolution product and involution to all of $\mathbb{C}[G]$ turns $\mathbb{C}[G]$ into a $*$-algebra over $\mathbb{C}$ with identity δ_{id_G}. For $f, g \in \mathbb{C}[G]$ and $x \in G$, we have

$$(f * g)(x) = \sum_{y \in G} f(y)\, g(y^{-1}x) \qquad \text{and} \qquad f^*(x) = \overline{f(x^{-1})}.$$

Unitary representations

We can "complete" the $*$-algebra $\mathbb{C}[G]$ with respect to various C*-norms to obtain C*-algebras. To do this, we make use of unitary representations of G.

Definition

Let A be a $*$-algebra with identity 1_A. The set

$$\mathcal{U}(A) := \{u \in A \mid u^*u = uu^* = 1_A\}$$

is a subgroup of A, and we call elements of $\mathcal{U}(A)$ **unitaries**.

A **unitary representation** of a group G in A is a homomorphism $u \colon G \to \mathcal{U}(A)$.

We have $\delta_x \in \mathcal{U}(\mathbb{C}[G])$ for each $x \in G$, so $x \mapsto \delta_x$ is a unitary representation of G in $\mathbb{C}[G]$.

The full group C*-algebra

If $u\colon G \to \mathcal{U}(A)$ is a unitary representation of G in a unital C*-algebra A, then the map $\delta_x \mapsto u_x$ extends linearly to a unital **$*$-representation** $\pi_u\colon \mathbb{C}[G] \to A$.

The **full norm** on $\mathbb{C}[G]$ is the C*-norm

$$\|f\| \coloneqq \sup\big\{\|\pi_u(f)\|_A \mid u\colon G \to \mathcal{U}(A) \text{ is a unitary representation}\big\}.$$

Completing $\mathbb{C}[G]$ with respect to this norm gives the **full group C*-algebra** $C^*(G)$.

The reduced group C*-algebra

The **left-regular representation** $\lambda\colon G \to \mathcal{U}\big(B(\ell^2(G))\big)$ characterised by $\lambda_x(\delta_y) \coloneqq \delta_{xy}$ for $x, y \in G$ induces a faithful (i.e. injective) representation $\pi_\lambda\colon \mathbb{C}[G] \to B(\ell^2(G))$.

The **reduced norm** on $\mathbb{C}[G]$ is the C*-norm given by $\|f\|_r \coloneqq \|\pi_\lambda(f)\|_{op}$. Completing $\mathbb{C}[G]$ with respect to this norm gives the **reduced group C*-algebra** $C^*_r(G)$.

Definition

A discrete group G is **amenable** if it admits a finitely additive left-invariant probability measure.

Theorem (Hulanicki 1966)

A discrete group G is amenable if and only if $C^*(G) = C^*_r(G)$.

Topological groupoids

We call a groupoid $\mathcal{G}$ a **topological groupoid** if it has a topology with respect to which multiplication and inversion are continuous.

Throughout, $\mathcal{G}$ will be a **locally compact Hausdorff** groupoid.

We call an open subset B of $\mathcal{G}$ an **open bisection** if $\mathbf{r}|_B$ and $\mathbf{s}|_B$ are homeomorphisms onto open subsets of $\mathcal{G}$.

We say that $\mathcal{G}$ is **étale** if it has a **basis of open bisections**.

We say that $\mathcal{G}$ is **ample** if it has a **basis of *compact* open bisections** (called **"cobs"**).

The full groupoid C*-algebra

Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. The collection $C_c(\mathcal{G})$ of compactly supported $\mathbb{C}$-valued functions on $\mathcal{G}$ is a vector space under pointwise operations.

We define a convolution product and involution on $C_c(\mathcal{G})$ by

$$(f * g)(\alpha) \coloneqq \sum_{\beta \in \mathbf{r}(\alpha)\mathcal{G}} f(\beta)\, g(\beta^{-1}\alpha) \qquad \text{and} \qquad f^*(\alpha) \coloneqq \overline{f(\alpha^{-1})}.$$

Under these operations, $C_c(\mathcal{G})$ is a $*$-algebra. The **full groupoid C*-algebra** $C^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to the **full norm**, which is given by

$$\|f\| \coloneqq \sup\big\{\|\pi(f)\| \mid \pi \text{ is a } *\text{-representation of } C_c(\mathcal{G})\big\}.$$

The reduced groupoid C*-algebra

For each $x \in \mathcal{G}^{(0)}$, define $\pi_x : C_c(\mathcal{G}) \to B\big(\ell^2(\mathcal{G}x)\big)$ by $\pi_x(f)g = f * g$ for all $f \in C_c(\mathcal{G})$ and $g \in \ell^2(\mathcal{G}x)$. Each π_x is a $*$-representation, called a **left-regular representation** of $C_c(\mathcal{G})$.

The **reduced groupoid C*-algebra** $C_r^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to the **reduced norm**, which is given by

$$\|f\|_r \coloneqq \sup\Big\{\|\pi_x(f)\|_{\mathrm{op}} \mid x \in \mathcal{G}^{(0)}\Big\}.$$

Theorem (Renault 1980 & Willett 2015)

If $\mathcal{G}$ is an amenable Hausdorff étale groupoid, then $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$. The converse is false.

Steinberg algebras

Let $\mathcal{G}$ be an **ample** Hausdorff groupoid, and let R be a commutative unital ring. Define

$$A_R(\mathcal{G}) := \operatorname{span}_R\left\{1_B : G \to R \,:\, B \subseteq \mathcal{G} \text{ is a cob}\right\} = \{f \in C_c(\mathcal{G}, R) : f \text{ is locally constant}\}.$$

Then $A_R(\mathcal{G})$ is an R-module under pointwise operations. For all cobs $X, Y \subseteq \mathcal{G}$, define

$$1_X * 1_Y = 1_{XY} \qquad \text{and} \qquad 1_X^* = 1_{X^{-1}}.$$

Extending this convolution product and involution to all of $A_R(\mathcal{G})$ turns $A_R(\mathcal{G})$ into a $*$-algebra over R, called the **Steinberg algebra** of $\mathcal{G}$.

Let $R = \mathbb{C}_d$, the complex numbers with the discrete topology. Then $A_{\mathbb{C}_d}(\mathcal{G})$ is a $*$-subalgebra of $C_c(\mathcal{G})$, and completing $A_{\mathbb{C}_d}(\mathcal{G})$ with respect to the full/reduced norm gives the full/reduced groupoid C*-algebra of $\mathcal{G}$.

Questions (next lecture)

- What is a **twisted groupoid**?
- **Structure theory:** What does the underlying groupoid tell us about the structure of the associated C*-algebras or Steinberg algebra?
- **Reconstruction theory:** Which C*-algebras or $*$-algebras can be built from twisted groupoids?

References

[1] J. Belk, *Puzzles, Groups, and Groupoids*, 2008, `cornellmath.wordpress.com/2008/01/27/puzzles-groups-and-groupoids` (accessed on 3rd January 2026).

[2] L.O. Clark, C. Farthing, A. Sims, and M. Tomforde, *A groupoid generalization of Leavitt path algebras*, Semigroup Forum **89** (2014), 501–517.

[3] A. Hulanicki, *Means and Følner condition on locally compact groups*, Studia Mathematica **27** (1966), 81–104.

[4] J. Renault, *A groupoid approach to C*-algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, New York, 1980.

[5] B. Steinberg, *A groupoid approach to discrete inverse semigroup algebras*, Adv. Math. **223** (2010), 689–727.

[6] R. Willett, *A non-amenable groupoid whose maximal and reduced C*-algebras are the same*, Münster J. Math. **8** (2015), 241–252.