Symmetries in noncommuting variables: **in** and out

Part One

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NZMRI 2026

The plan

- Connecting tissue for the two talks
- Joint similarity problem: old and new
- Separating invariants
- ToDo

Functions in several operator variables

$$f(X, Y) = X^{-1/2} YXYX^{-1/2} - Y^2$$

for $n \times n$ matrices X, Y, or operators X, Y

- analysis
- geometry
- optimisation
- probability
- dependence on n
- ...

Main source of challenges / excitement:

noncommuting variables dimensionless phenomena

Symmetries and invariants in nc variables

two types

(1) symmetries of arguments

(X, Y) and (SXS^{-1}, SYS^{-1}) represent the same pair of operators

general linear / unitary group acting on tuples of matrices by conjugation

$\mathrm{GL}_n, \mathrm{U}_n \curvearrowright \mathrm{M}_n(\mathbb{C})^2$

(2) symmetries of functions

$f(X, Y) = X^2 - XY - YX + Y^2$ is symmetric under $X \leftrightarrow Y$

finite group acting on suitable functions/expressions in x, y

$G \curvearrowright \mathbb{C}\langle\!\langle x, y \rangle\!\rangle$

Similarity

students' "favourite" topic

$X, Y \in \mathrm{M}_m(\mathbb{C})$ are similar if $Y = SXS^{-1}$ for some $S \in \mathrm{GL}_m(\mathbb{C})$

- Jordan canonical form
- Weyr characteristic: X, Y similar iff

$$\mathrm{rk}(X - \lambda I)^r = \mathrm{rk}(Y - \lambda I)^r \qquad \forall \lambda \in \mathbb{C}, r \in \mathbb{N}$$

functions $X \mapsto \mathrm{rk}(X - \lambda I)^r$ are separating invariants

Joint similarity

n-tuples of matrices

$X, Y \in \mathrm{M}_m(\mathbb{C})^n$ are (jointly) similar if for some $S \in \mathrm{GL}_m(\mathbb{C})$,

$$Y_j = SX_jS^{-1} \qquad \text{for } j = 1, \dots, n$$

Shortly: $Y = SXS^{-1}$, $X \sim Y$

Operator theory, representation theory, invariant theory ...
want to understand matrix tuples up to similarity

The easy and the hard aspect of similarity

Easy: given X and Y, decide whether they are are similar

Probabilistic: solve the linear system $Y_1 S = S X_1, \ldots, Y_n S = S X_n$ in S, check if the generic solution is invertible

Deterministic: isomorphism of modules in polynomial time

The easy and the hard aspect of similarity

Easy: given X and Y, decide whether they are are similar

Probabilistic: solve the linear system $Y_1 S = S X_1, \ldots, Y_n S = S X_n$ in S, check if the generic solution is invertible

Deterministic: isomorphism of modules in polynomial time

Hard: canonical form ?? $(n \geq 2)$

Hopeless! Drozd[77], LeBruyn[97]

classification of matrix tuples up to similarity is a wild problem

Friedland[83]: iterative procedure à la quantifier elimination

"up to finitely many exceptions"

Gelfand reduction

the "simplest" version of a wild problem

Gelfand-Ponomarev[69], Nathanson[80]:

Classification of n-tuples of matrices up to $\sim$, for $n \geq 2$

$$\Longleftrightarrow$$

Classification of pairs of commuting jointly 3-nilpotents up to $\sim$

X_1, X_2 such that $X_1 X_2 = X_2 X_1$ and $X_1^3 = X_1^2 X_2 = X_1 X_2^2 = X_2^3 = 0$

reps of the 6-dim commutative algebra $\mathbb{C}[x_1, x_2]/(x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3)$

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pairs of commuting jointly 2-nilpotent matrices: tame Šivic

The middle ground - separating invariants

Task: find a natural collection of separating invariants $\{f_\alpha\}_\alpha$,

$$X \sim Y \iff f_\alpha(X) = f_\alpha(Y) \quad \forall \alpha$$

E.g., for $n = 1$ take $f_{\lambda,r}(X) = \mathrm{rk}(X - \lambda I)^r$

Joint unitary similarity

for comparison

$X, Y \in \mathrm{M}_m(\mathbb{C})^n$ are unitarily similar if there is a unitary $S \in \mathrm{U}_m(\mathbb{C})$ such that $Y = SXS^*$

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Wiegmann[65], Procesi[76]:
X, Y are unitarily similar iff

$$\operatorname{tr} w(X, X^*) = \operatorname{tr} w(Y, Y^*)$$

for all products $w = w(x_1, \dots, x_n, x_1^*, \dots, x_n^*)$ of length $\leq m^2$

Closed/non-closed orbits

$\mathrm{U}_m(\mathbb{C})$ is a compact group, its orbits in $\mathrm{M}_m(\mathbb{C})^n$ are closed, $\overline{X^{\mathrm{U}_m}} = X^{\mathrm{U}_m}$

Orbits of $\mathrm{GL}_m(\mathbb{C})$ are not closed in general,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \lim_{t\to\infty} \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \lim_{t\to\infty} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^{-1},$$

so $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \notin \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^{\mathrm{GL}_2}$ but $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \overline{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^{\mathrm{GL}_2}}$

Closedness and smoothness

$X \in \mathrm{M}_m(\mathbb{C})^n$ is irreducible if $X_1, \ldots, X_n$ don't have a common invariant subspace (Burnside: they generate $\mathrm{M}_m(\mathbb{C})$ as an algebra)

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Artin[69]: X^{GL_m} is closed iff X is a direct sum of irreducibles; each orbit contains a unique closed orbit, $\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix} \in \overline{\begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}^{\mathrm{GL}}}$

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LeBruyn-Procesi[06]: closed GL_m-orbits in $\mathrm{M}_m(\mathbb{C})$ form a variety, its smooth points are precisely the orbits of irreducibles

Separating orbit closures

$\overline{X^{\mathrm{GL}}} \cap \overline{Y^{\mathrm{GL}}} \neq \emptyset$: traces of products — Procesi, Razmyslov

$X^{\mathrm{GL}} \subseteq \overline{Y^{\mathrm{GL}}}$: open! — (will get back to it)

$X^{\mathrm{GL}} = Y^{\mathrm{GL}}$: equivalent to $X^{\mathrm{GL}} \subseteq \overline{Y^{\mathrm{GL}}}$ and $Y^{\mathrm{GL}} \subseteq \overline{X^{\mathrm{GL}}}$;
can't hope for continuous separating invariants, but still...

Curto-Herrero conjecture

Noncommutative polynomials: $\mathbb{C}<x_1, \ldots, x_n>$

e.g. $x_1 x_2 x_1^2 - x_2 x_1 + x_1 x_2 - 3$

Curto-Herrero conjecture

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Conjecture (Curto-Herrero[85])

Let $X, Y \in \mathrm{M}_m(\mathbb{C})^n$*. Then* $X \sim Y$ *iff*

$$\operatorname{rk} f(X) = \operatorname{rk} f(Y)$$

for all $f \in \mathbb{C}<x_1, \ldots, x_n>$*.*

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- $f(SXS^{-1}) = Sf(X)S^{-1}$, so $\implies$ is the easy one
- true for $n = 1$ (JCF, Weyr characteristic)
- true for $m = 2$ Curto-Herrero
- true if X is a direct sum of irreducibles Klep-Helton-V[18]

Hadwin-Larson amendment

Curto-Herrero Conj is false:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad Y_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Let $X, Y \in \mathrm{M}_m(\mathbb{C})^n$*. Then* $X \sim Y$ *iff*

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for all square matrices F *over* $\mathbb{C}{<}x_1, \dots, x_n{>}$.

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- $F(SXS^{-1}) = (I \otimes S)F(X)(I \otimes S)^{-1}$, so $\implies$ is still easy
- for the above counterex and $F = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$,
  $\operatorname{rk} F(X) = 2 \neq 1 = \operatorname{rk} F(Y)$

Ranks of affine matrix pencils

Hadwin-Larson Conj is true

Affine matrices over $\mathbb{C}{<}x_1, \ldots, x_n{>}$ are called affine matrix pencils:

$F = A_0 + A_1 x_1 + \cdots + A_n x_n,$

$F(X) = A_0 \otimes I + A_1 \otimes x_1 + \cdots + A_n \otimes x_n$

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Theorem (Derksen-Klep-Makam-V[23])

Let $X, Y \in \mathrm{M}_m(\mathbb{C})^n$. Then $X \sim Y$ iff

$$\operatorname{rk} F(X) = \operatorname{rk} F(Y)$$

for all $mn \times mn$ affine matrix pencils F.

module extensions & degenerations: Auslander, Bongartz, Smalø...

Analogous for $\mathrm{U}_m(\mathbb{C})$, $\mathrm{O}_m(\mathbb{C})$, $\mathrm{Sp}_m(\mathbb{C})$, $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$

One-sided version?

Conjecture (Hadwin-Larson[03])

Let $X, Y \in \mathrm{M}_m(\mathbb{C})^n$. *Then* $X \in \overline{Y^{\mathrm{GL}_m}}$ *iff*

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... is false

An example

Carlson: $X, Y \in \mathrm{M}_4(\mathbb{C})^2$ with $X \notin \overline{Y^{\mathrm{GL}}}$ but $\mathrm{rk}\, F(X) \leq \mathrm{rk}\, F(Y)$

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If ζ_{ij}, ξ_{ij} denote the coordinates of $\mathrm{M}_4(\mathbb{C}) \times \mathrm{M}_4(\mathbb{C})$, then

$$p = \zeta_{43}\xi_{21} - \zeta_{41}\xi_{23} - \zeta_{23}\xi_{41} + \zeta_{21}\xi_{43}$$

satisfies $p(PYP^{-1}) = 0$ for all $P \in \mathrm{GL}_4(\mathbb{C})$, and $p(X) = 2$

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There is $P_\varepsilon \in \mathrm{GL}_5(\mathbb{C})$ such that $X \oplus 0 = \lim_{\varepsilon \to 0} P_\varepsilon (Y \oplus 0) P_\varepsilon^{-1}$.

$$P_\varepsilon = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&1&0&1 \\ 0&0&\varepsilon^2&0&0 \\ 0&0&0&\varepsilon^2&0 \\ 0&\varepsilon&-\varepsilon&0&0 \end{pmatrix}$$

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$$\operatorname{rk} F(X) = \operatorname{rk} F(X \oplus 0) - \operatorname{rk} F(0) \leq \operatorname{rk} F(Y \oplus 0) - \operatorname{rk} F(0) = \operatorname{rk} F(Y)$$

Moral & Final thoughts

- no multivariate JCF
- closed orbits have good geometry
- ranks of pencils separate orbits

Conjecture

TFAE for $X, Y \in \mathrm{M}_n(\mathbb{C})^m$*:*

(a) $X \oplus 0_k \in \overline{(Y \oplus 0_k)^{\mathrm{GL}_{n+k}}}$ *for some* $k \in \mathbb{N}$*;*

(b) $\operatorname{rk} F(X) \leq \operatorname{rk} F(Y)$ *for all affine pencils* F*.*

Problem

Certify $X \in / \notin \overline{Y^{\mathrm{GL}}}$ *with invariants in a natural way?*