An introduction to holomorphic rigidity

Peter Huxford, Rice University

Moduli spaces and holomorphic maps

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- Can we describe all holomorphic maps between given moduli spaces?
- Are they all given by some previously known algebraic construction?

Configuration spaces

$$\mathrm{UConf}_n X = \Big(\{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\} \Big) \Big/ S_n$$

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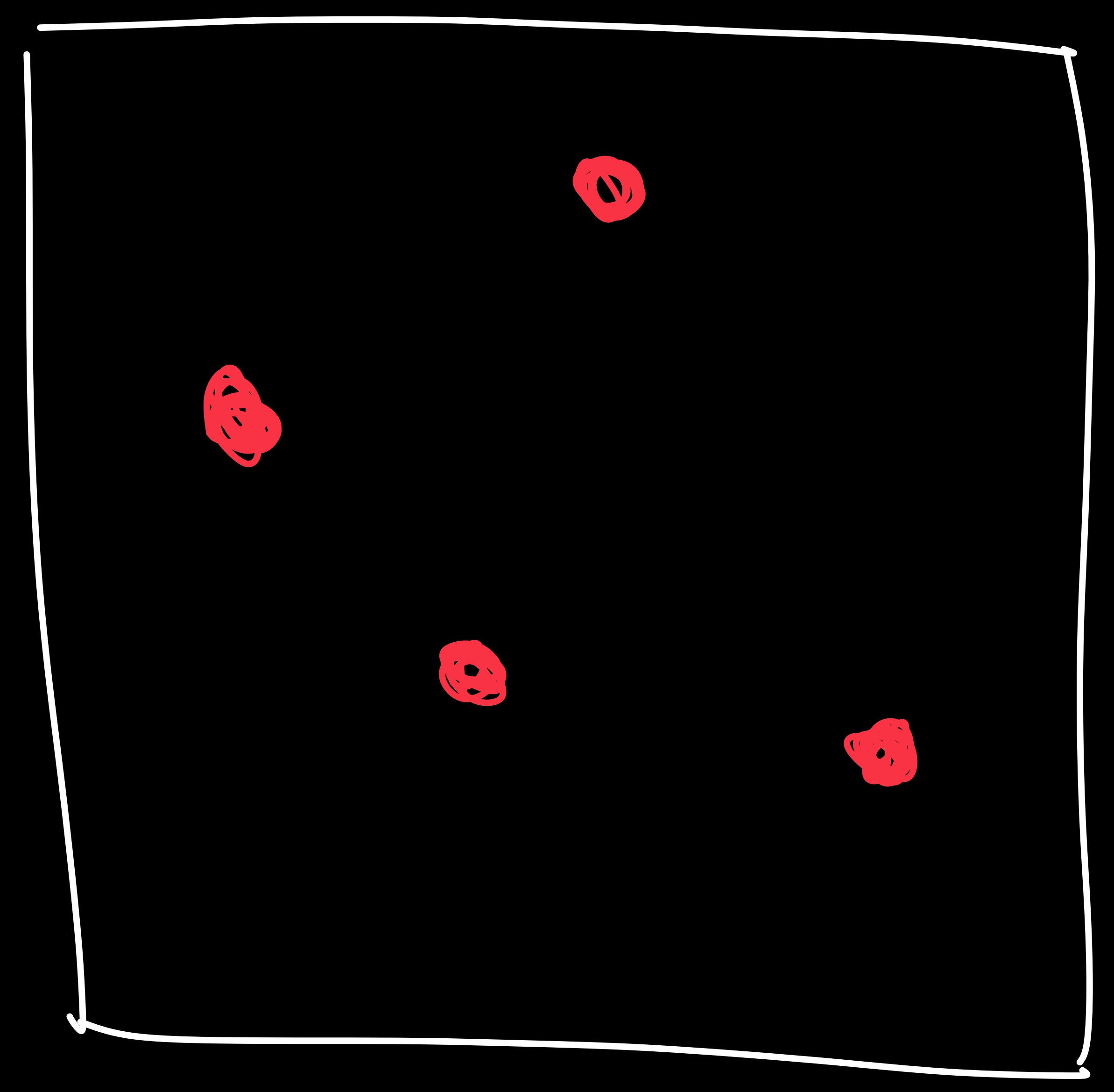

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$$R\colon \mathrm{UConf}_4\mathbb{C} \to \mathrm{UConf}_3\mathbb{C}$$
$$\{x_1, \ldots, x_4\} \mapsto \{x_1x_4 + x_2x_3,\ x_1x_3 + x_2x_4,\ x_1x_2 + x_3x_4\}$$

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homeomorphic to a genus *g* surface.

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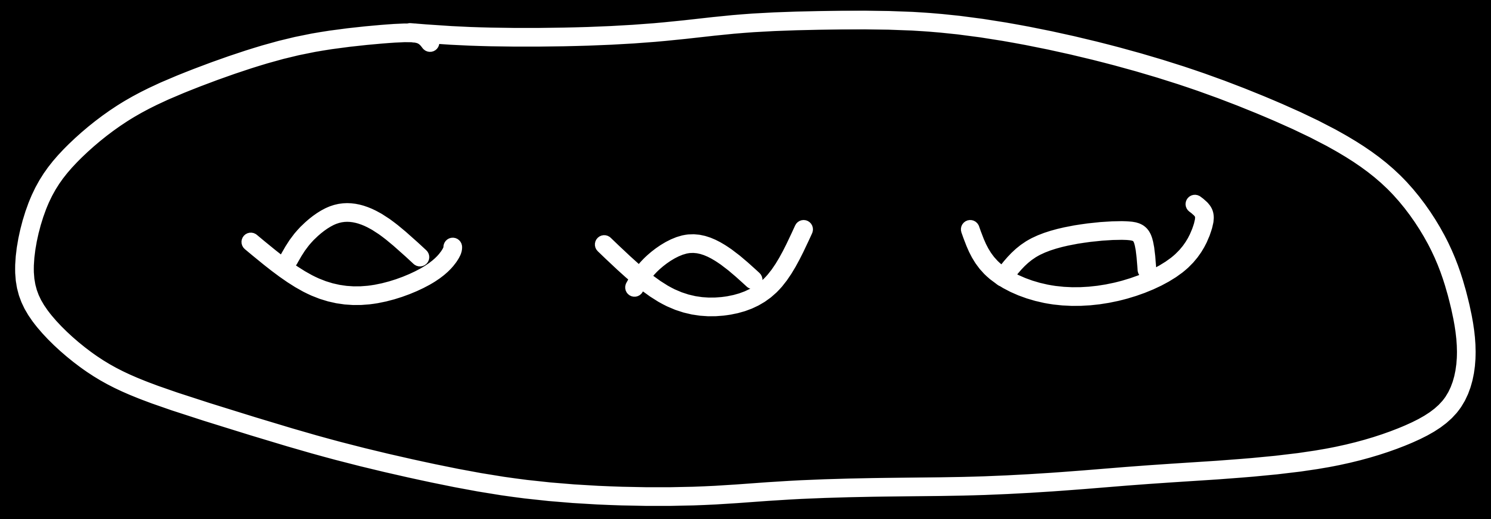

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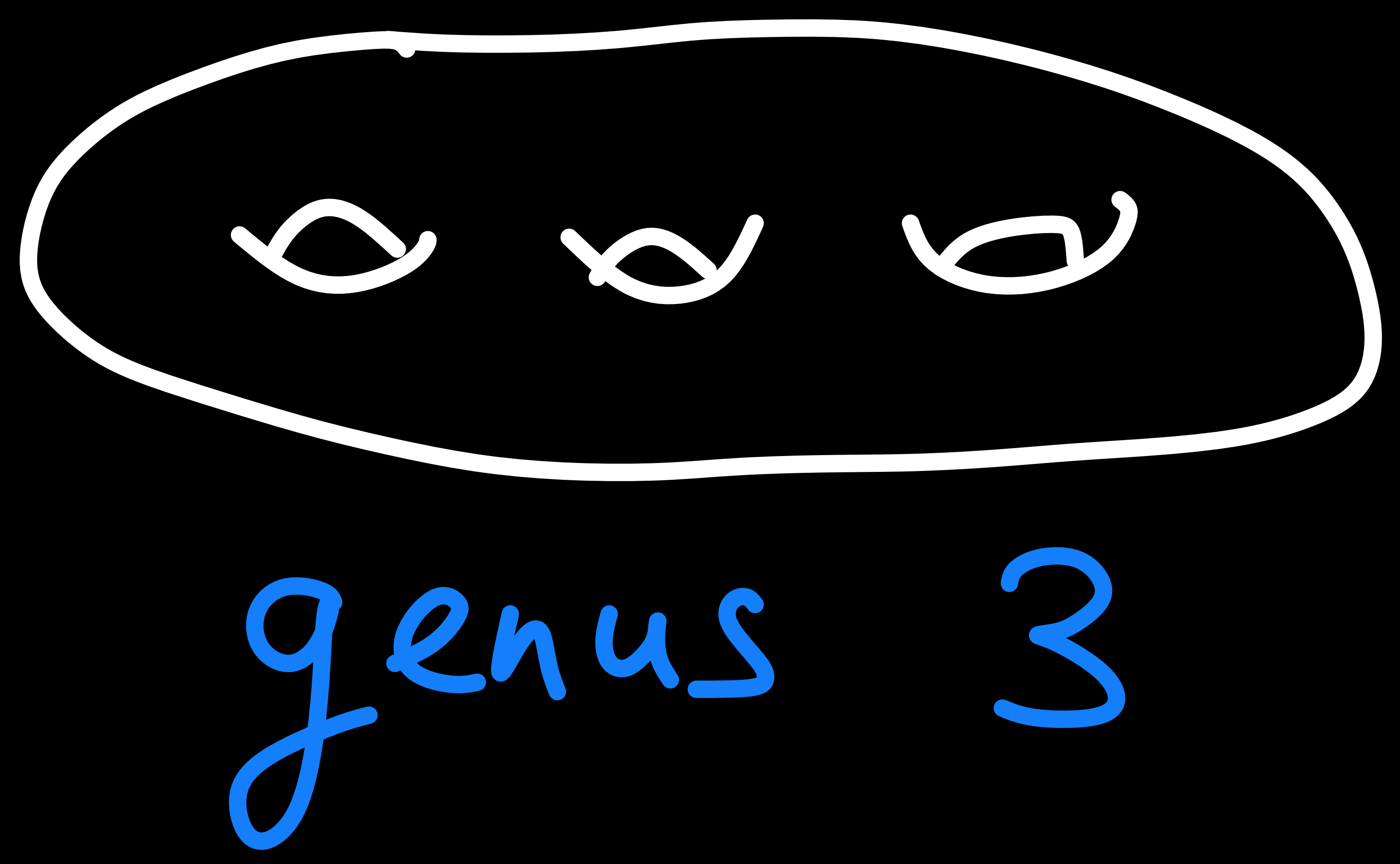

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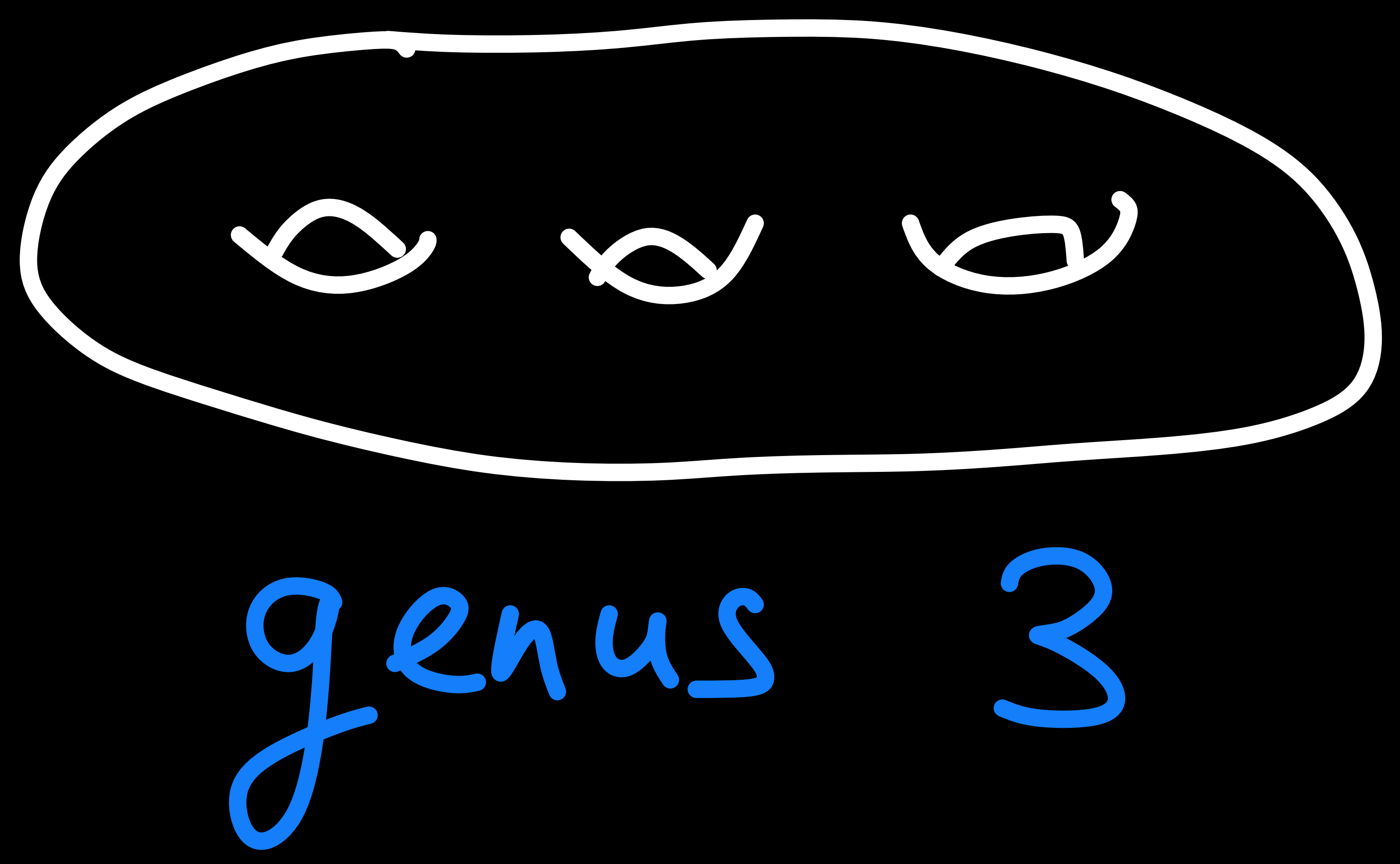

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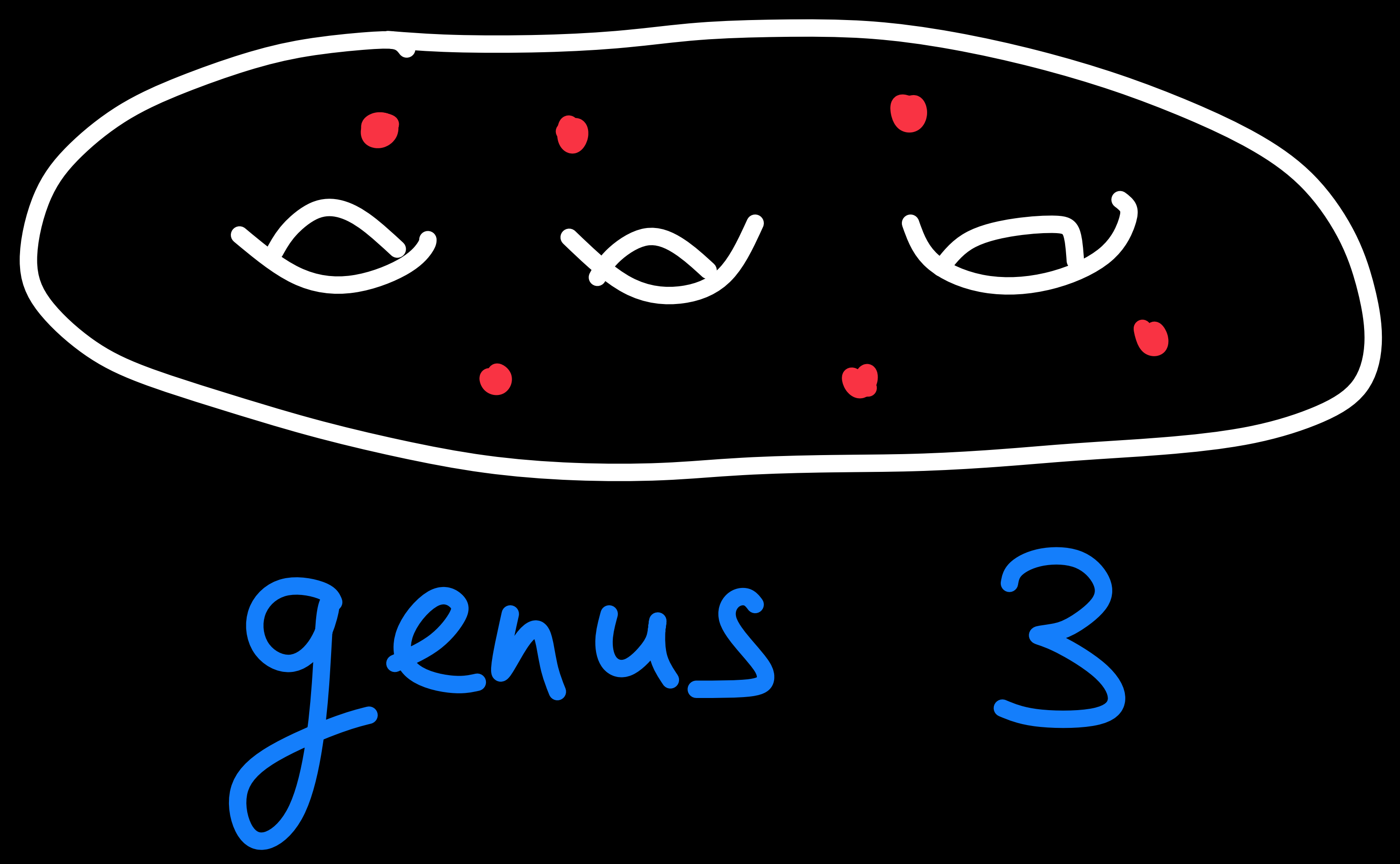

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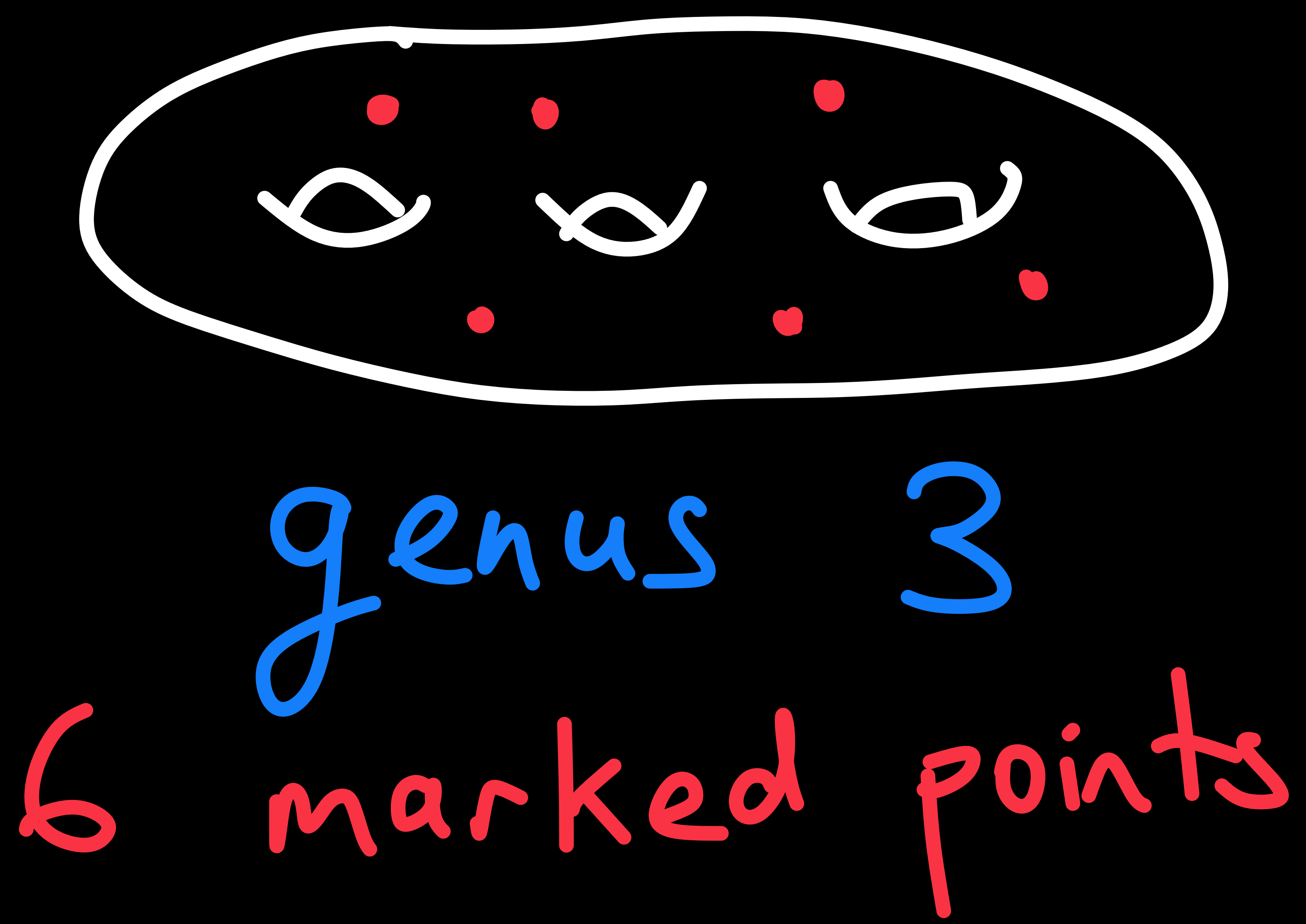

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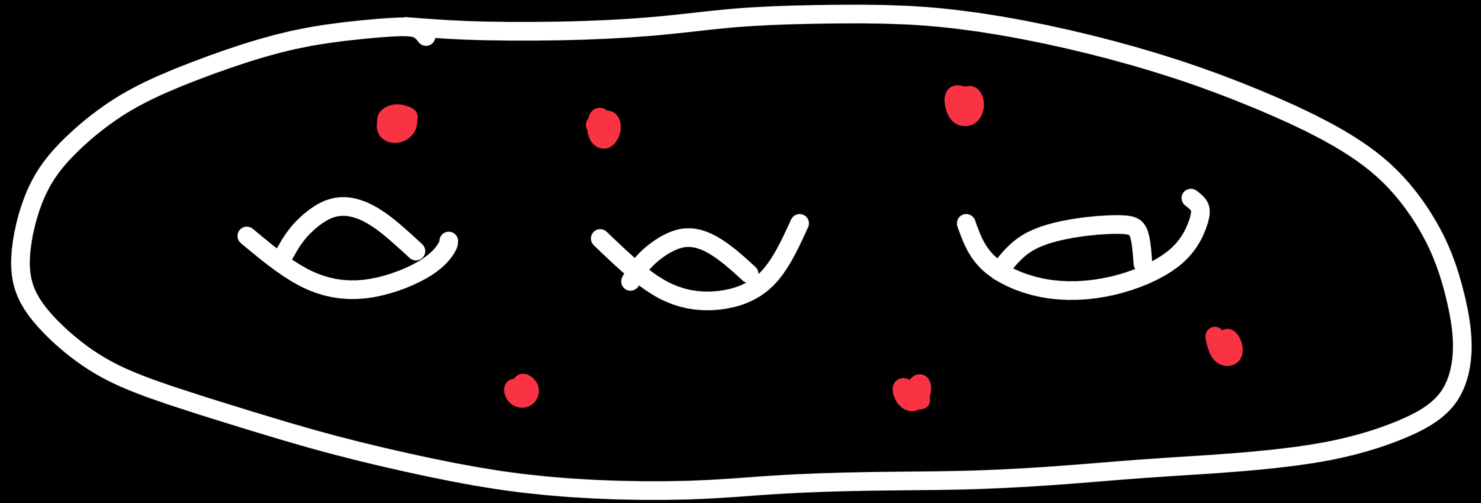

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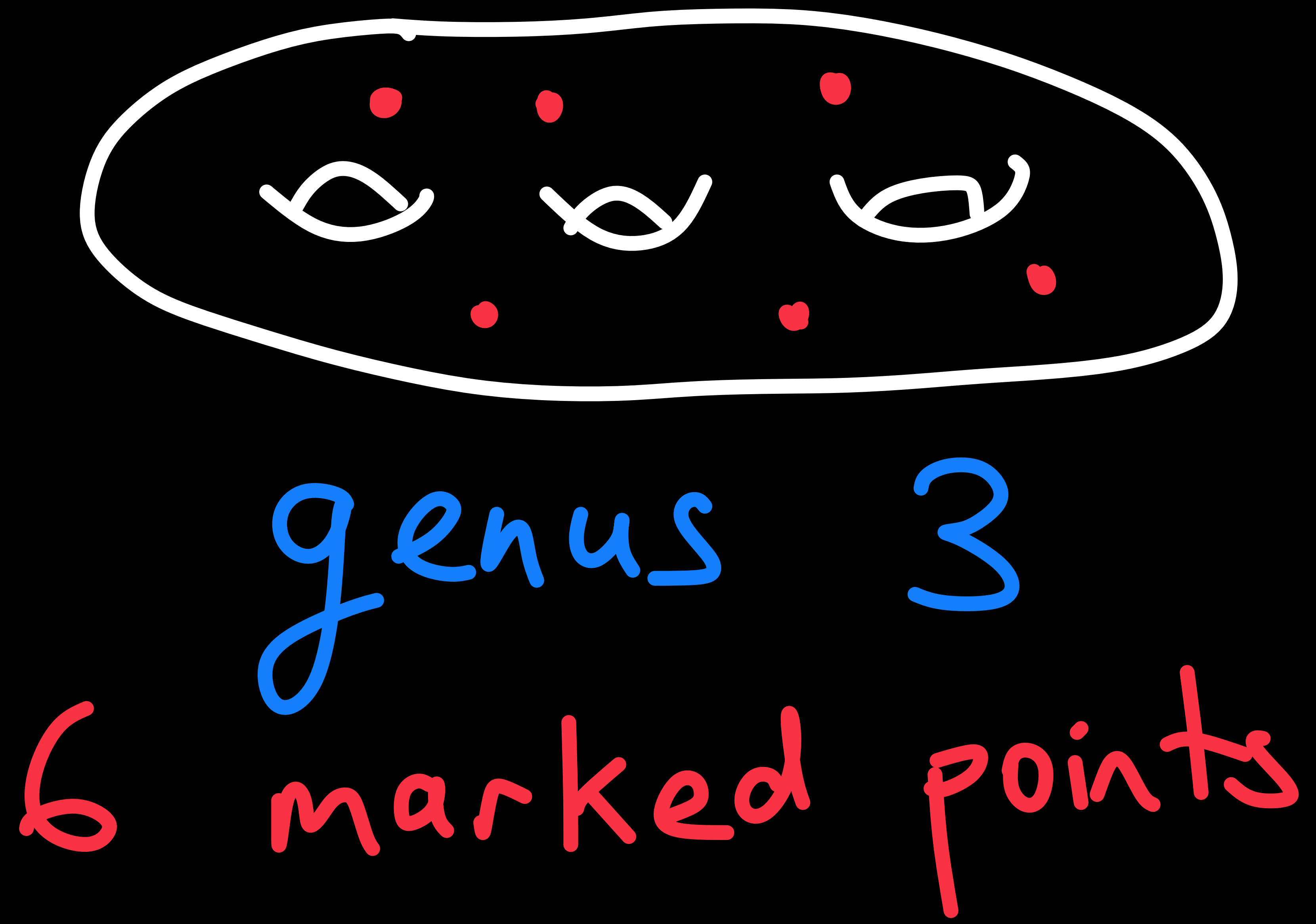

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We write $\mathcal{M}_{g,n} / S_n$ for the unordered variant, and also $\mathcal{M}_g = \mathcal{M}_{g,0}$.

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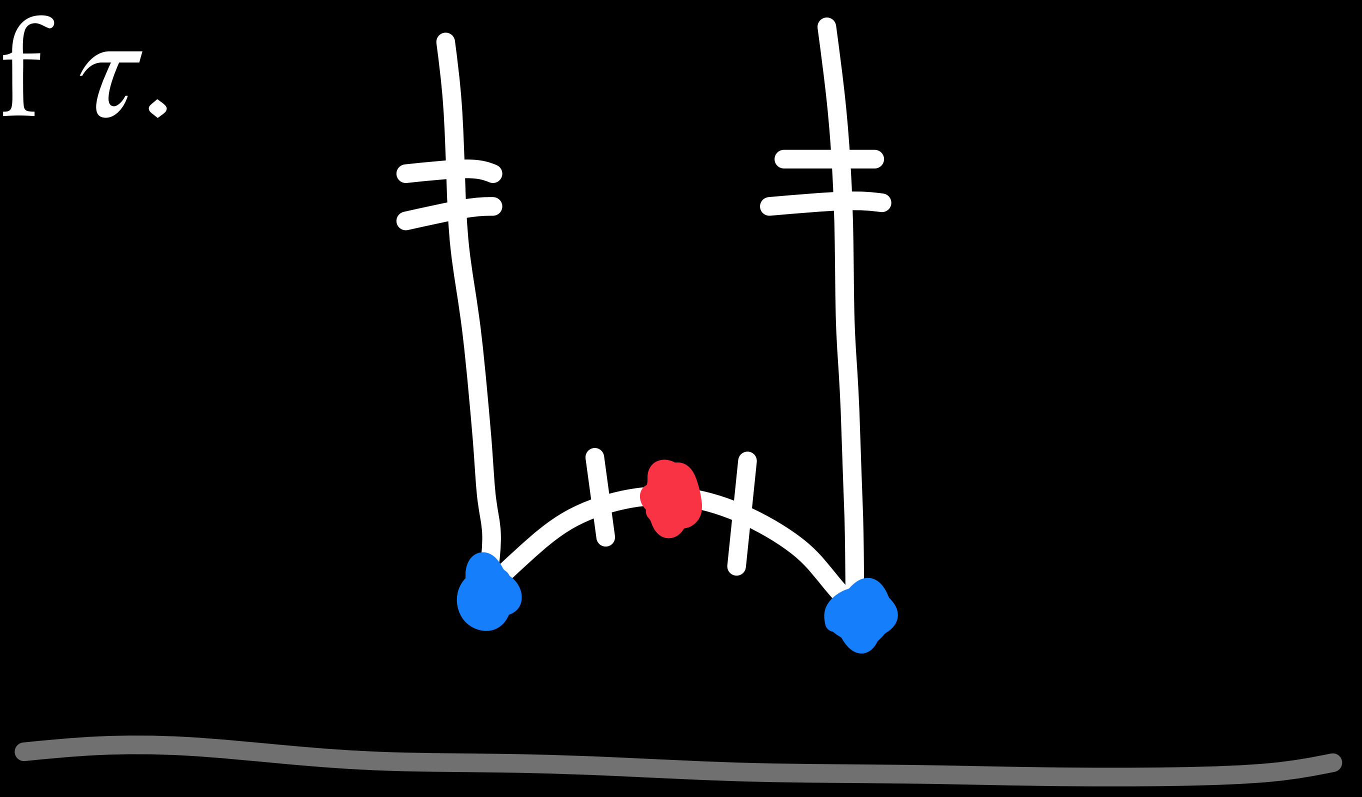

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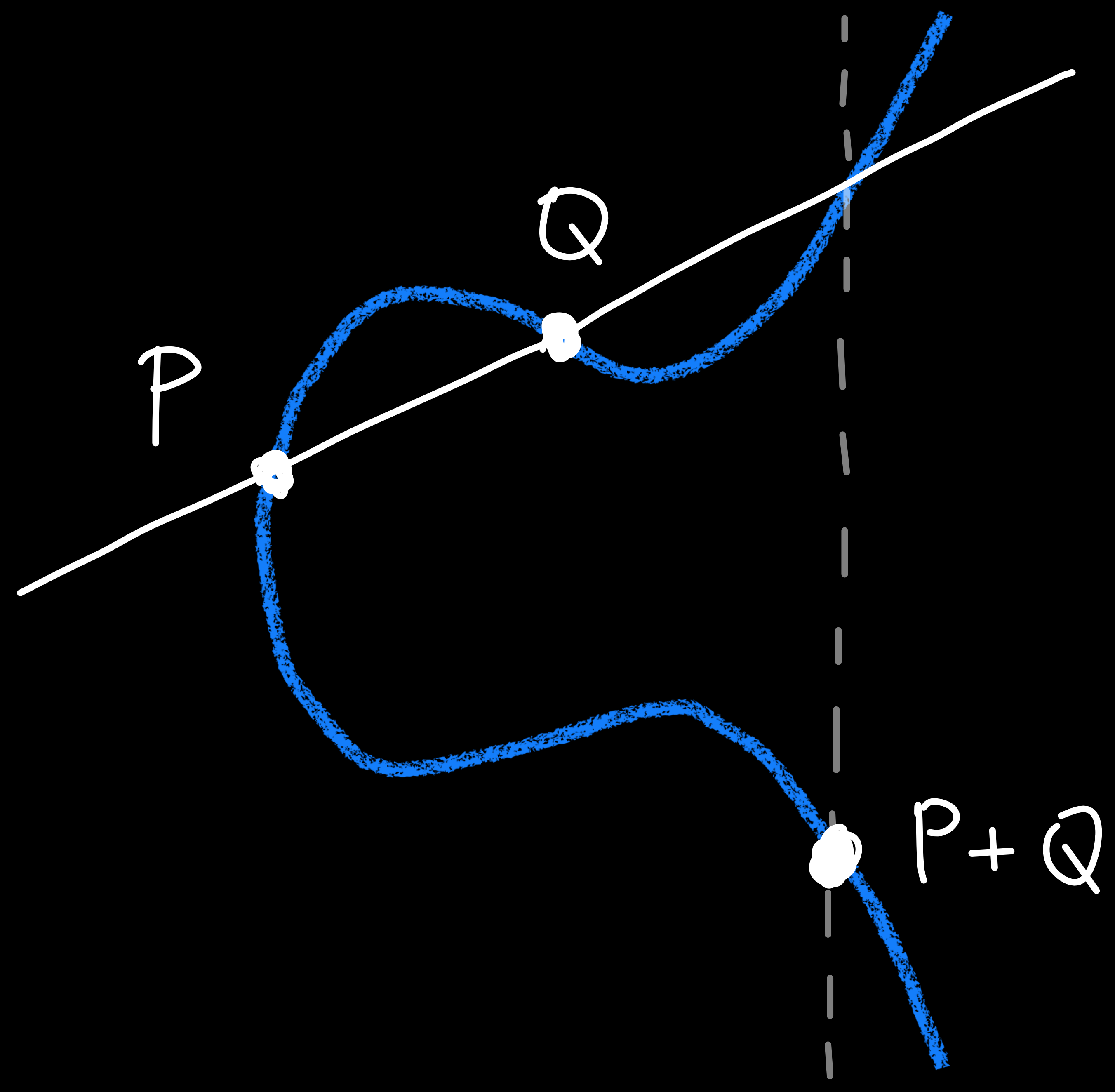

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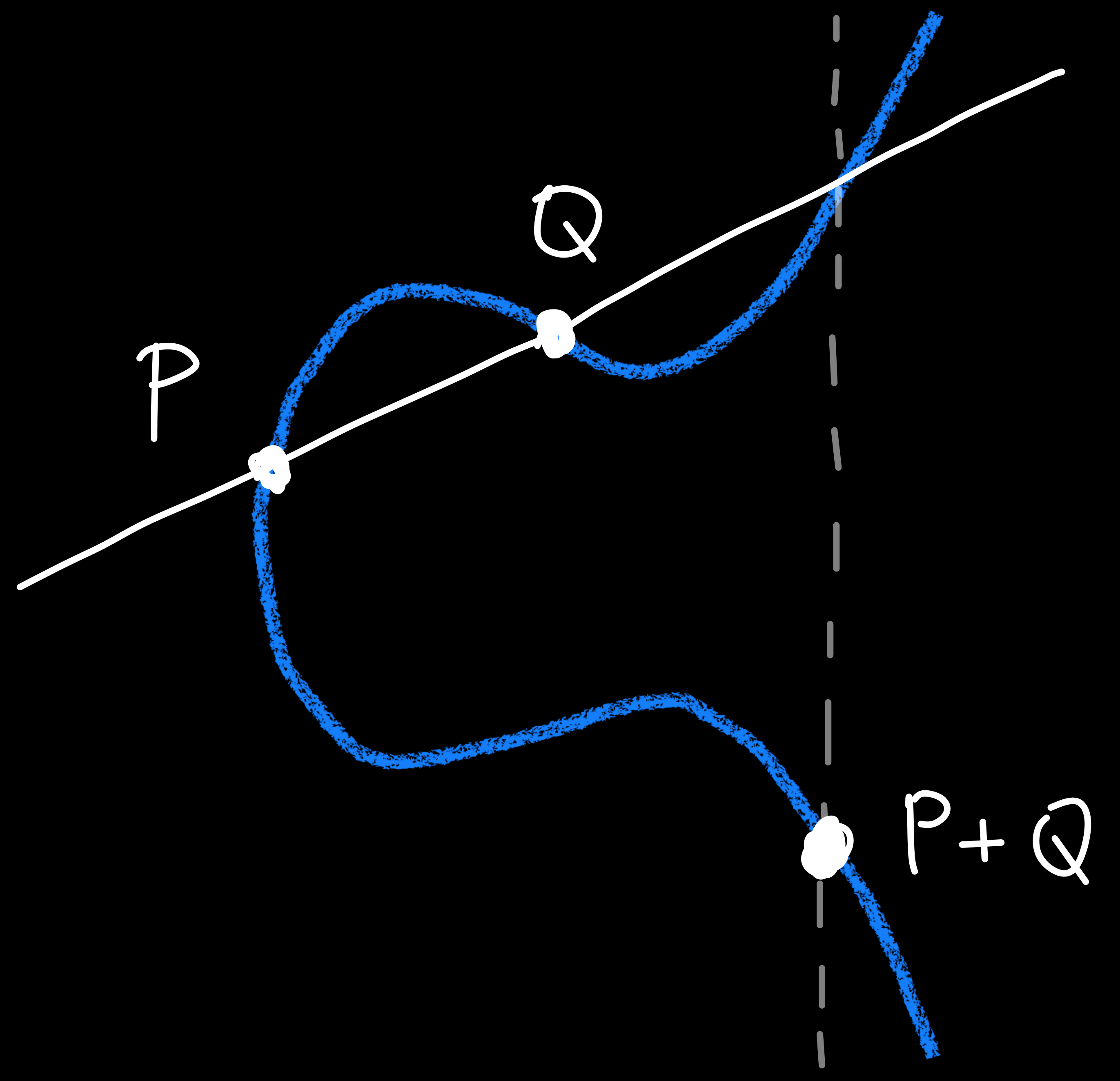

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This defines a holomorphic map $\mathcal{M}_g \to \mathcal{M}_h$ where $h = 2^{2g}(g-1)+1$.

Examples so far

Resolving the quartic: $\mathrm{UConf}_4\mathbb{C} \to \mathrm{UConf}_3\mathbb{C}$.

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Many constructions coming from other famous stories, e.g. the 27 lines on a smooth cubic surface, the Jacobian of a Riemann surface, …

Sample questions

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Each of the above are open in general, but have been solved in ranges.

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$$y^2 = (x - x_1)\cdots(x - x_n).$$

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If $f\colon \mathcal{M}_{g,n} \to \mathcal{M}_{h,m}$ is a non-constant holomorphic map, then $g = h$, $m \leq n$, and f is given by forgetting some marked points.

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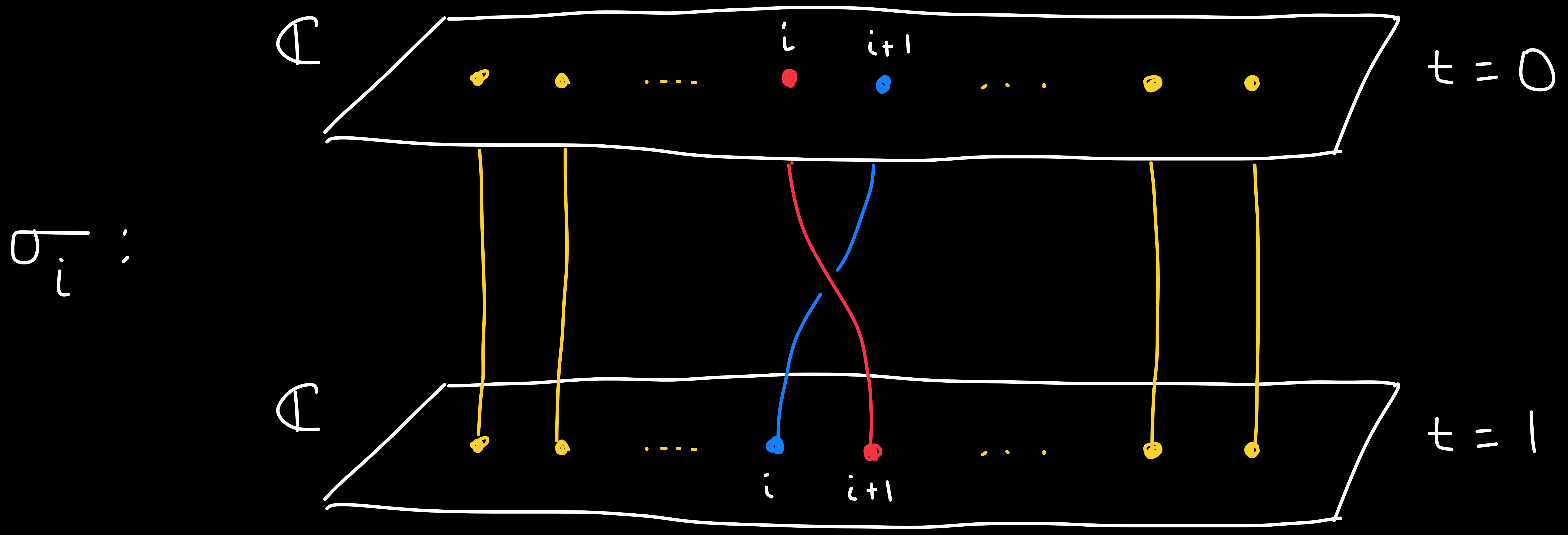
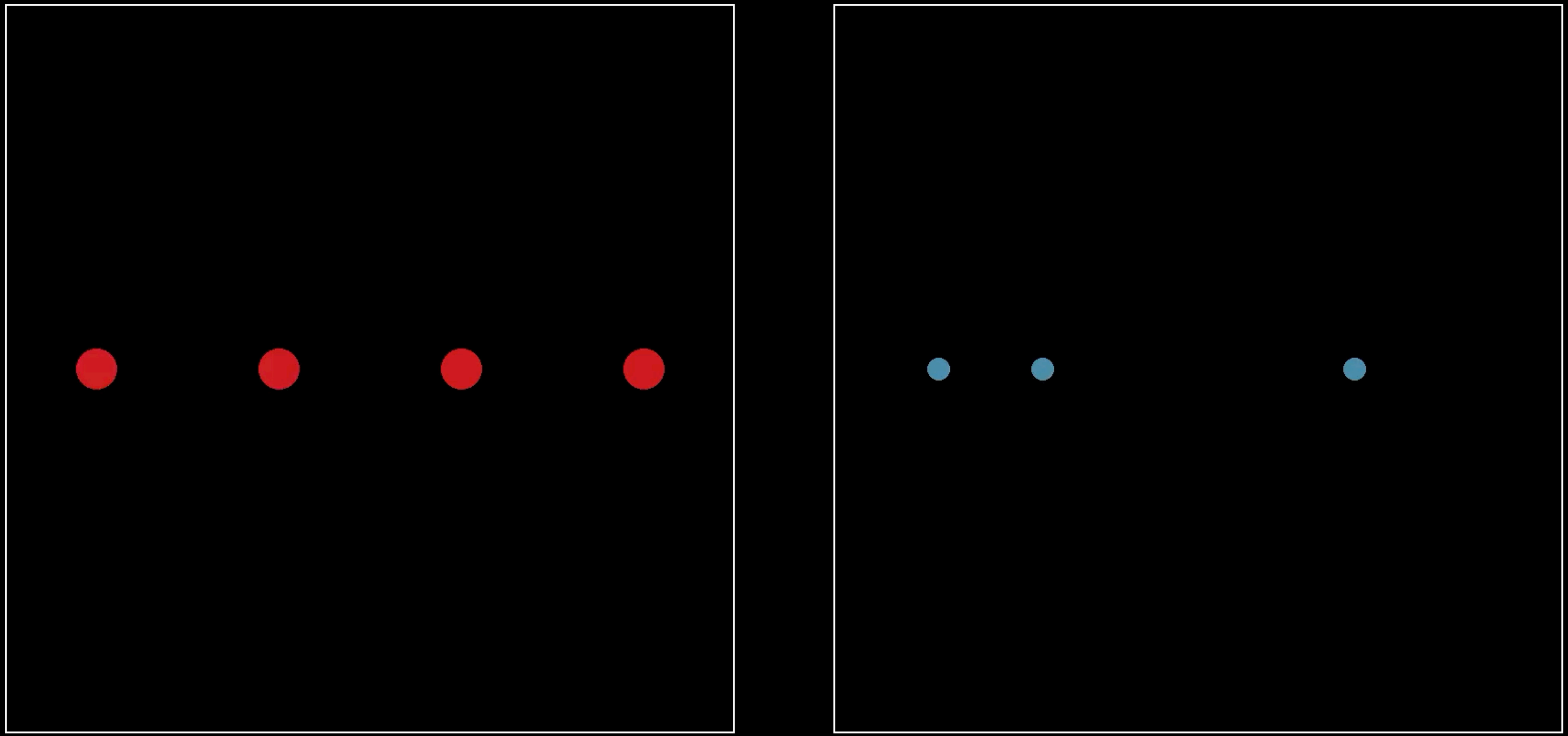
What are the homomorphisms $\pi_1(\mathscr{M}) \to \pi_1(\mathscr{N})$?

Braid groups

The fundamental group $\pi_1(\mathrm{UConf}_n\mathbb{C})$ is the *braid group* $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$.



Example: $R\colon \mathrm{UConf}_4\mathbb{C} \to \mathrm{UConf}_3\mathbb{C}$ induces $R_*\colon B_4 \to B_3$



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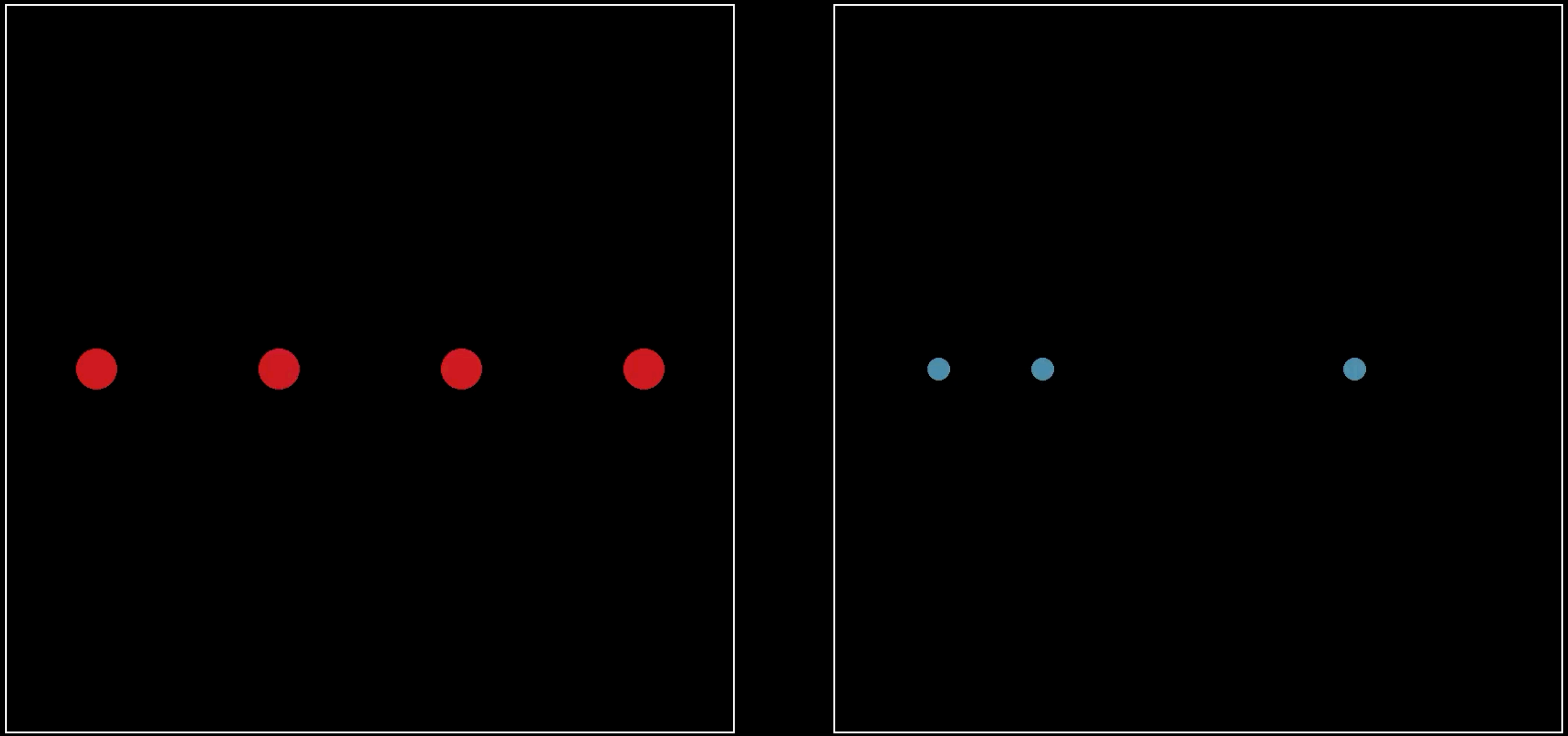

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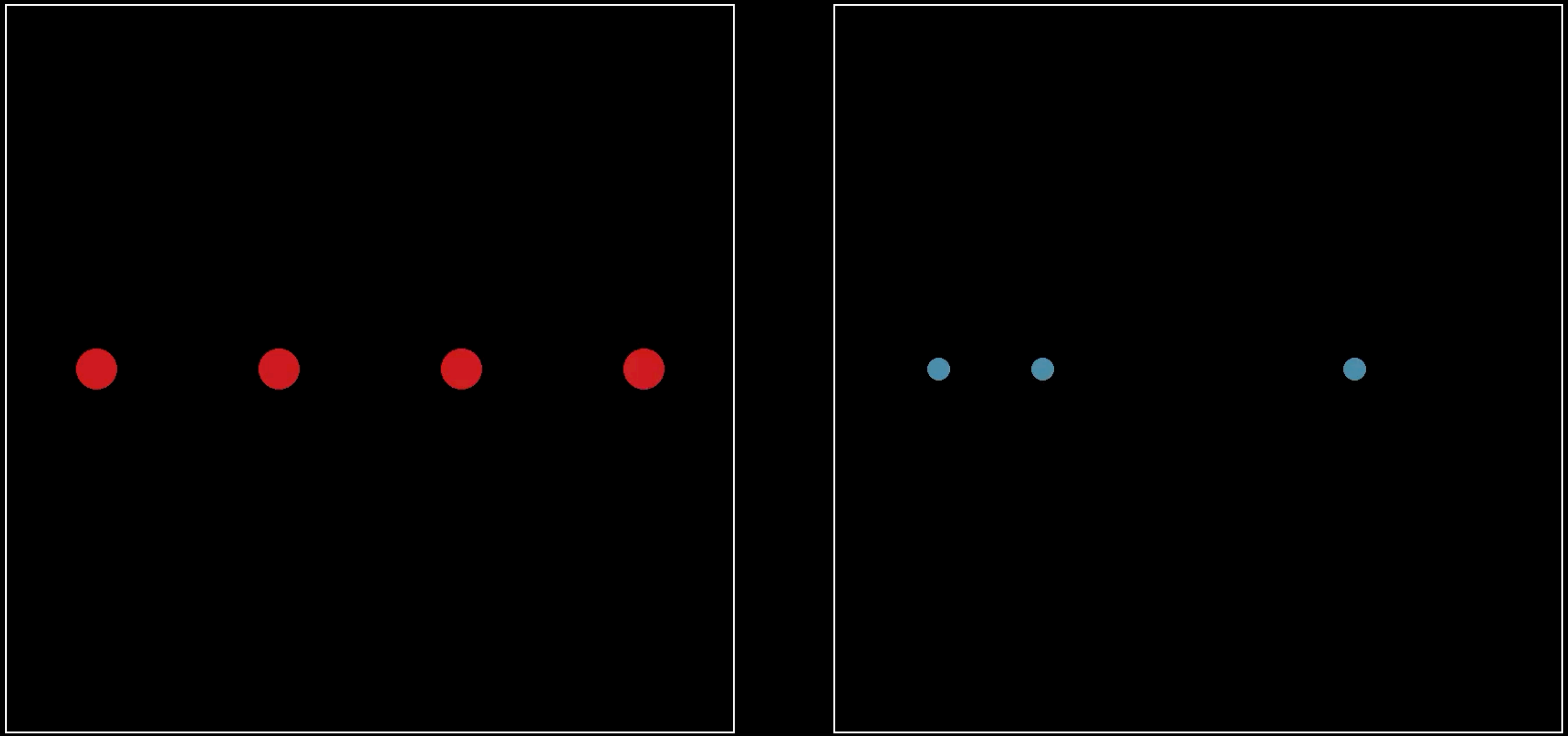

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Mapping class groups

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Special case: $\mathrm{Mod}_{1,1} \cong \mathrm{SL}_2\mathbb{Z}$.

Braid group as mapping class group

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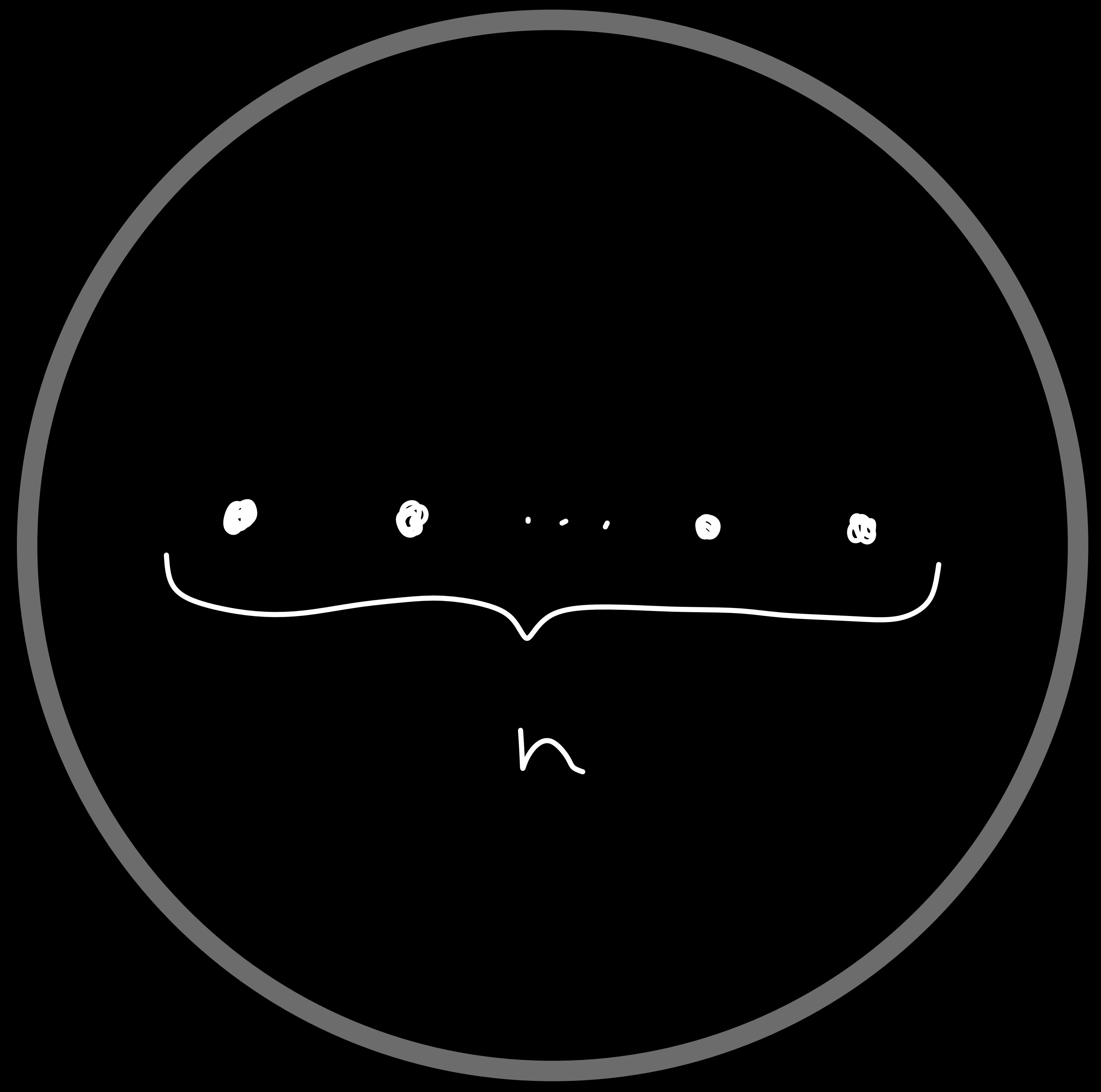

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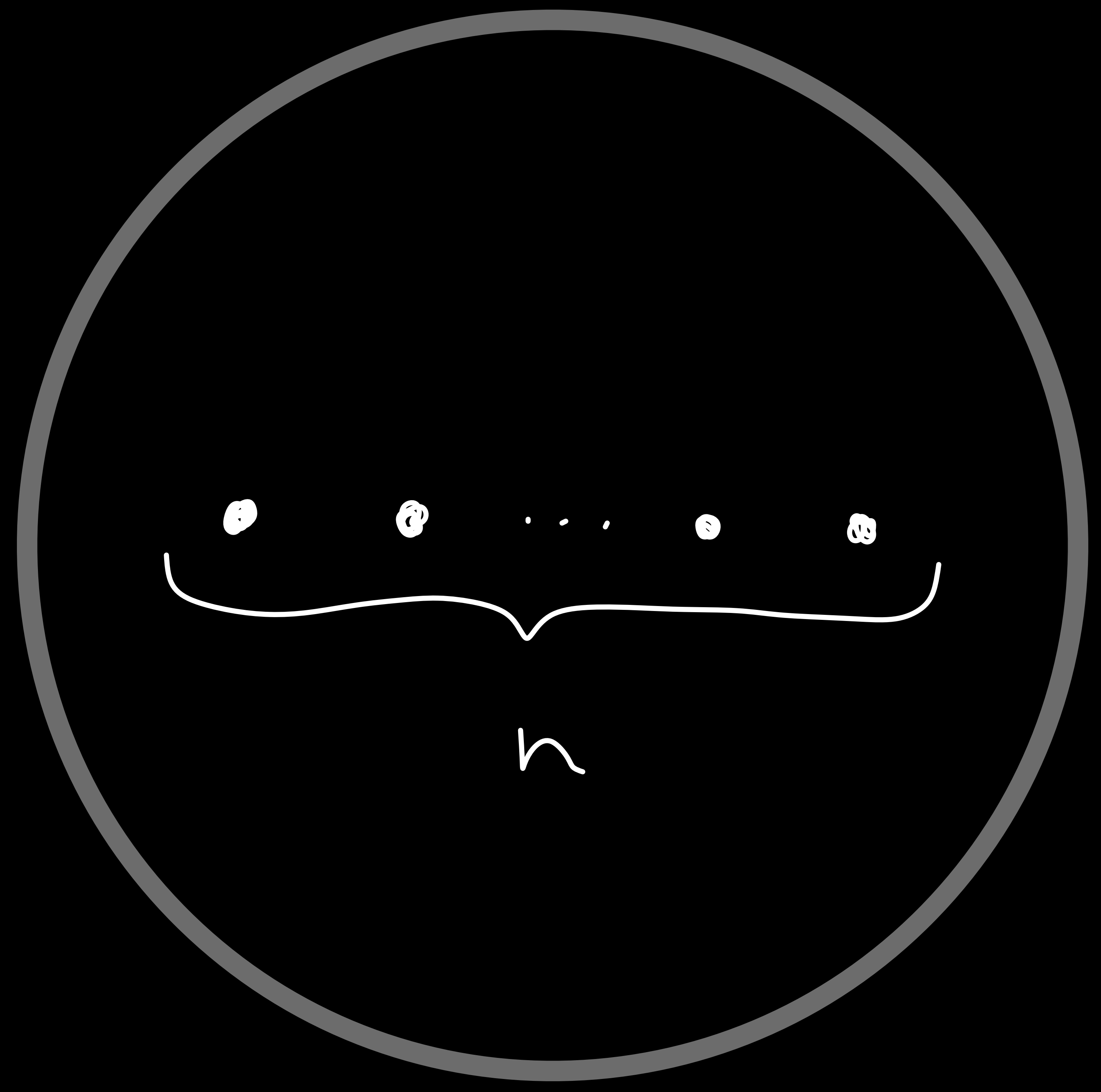
The natural map $B_n \to \mathrm{Mod}_{0,n}^1$ is an isomorphism.



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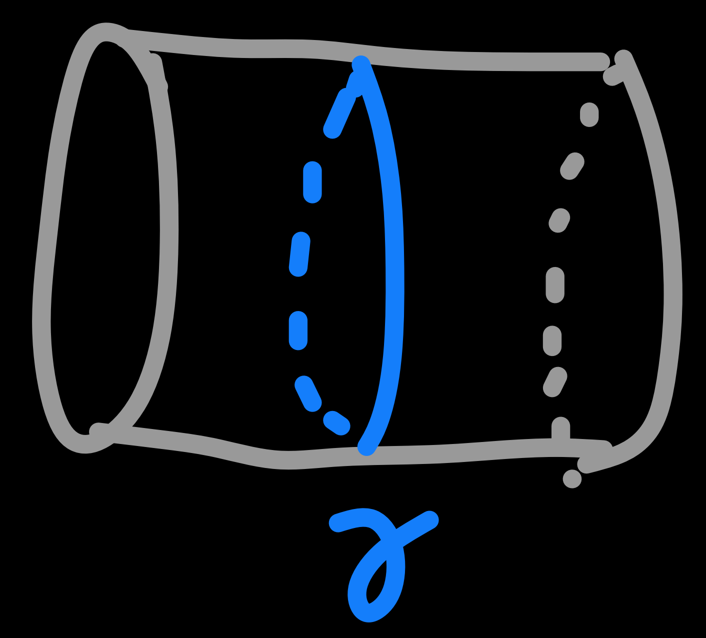

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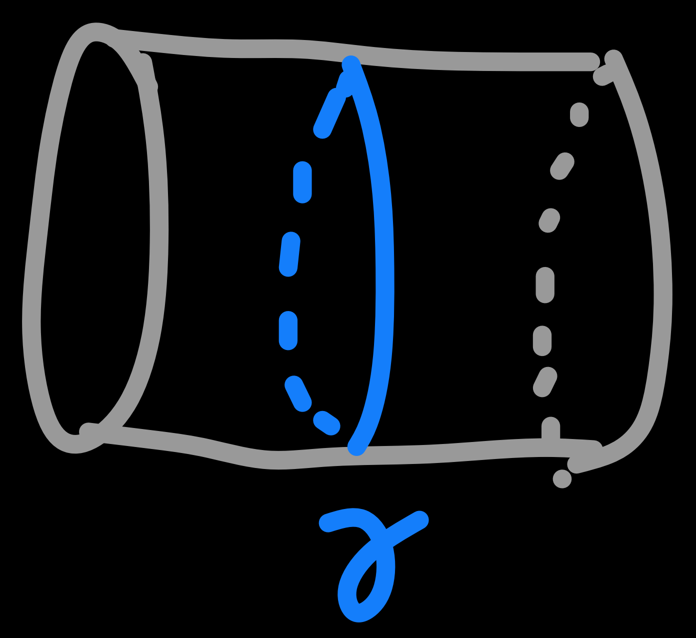

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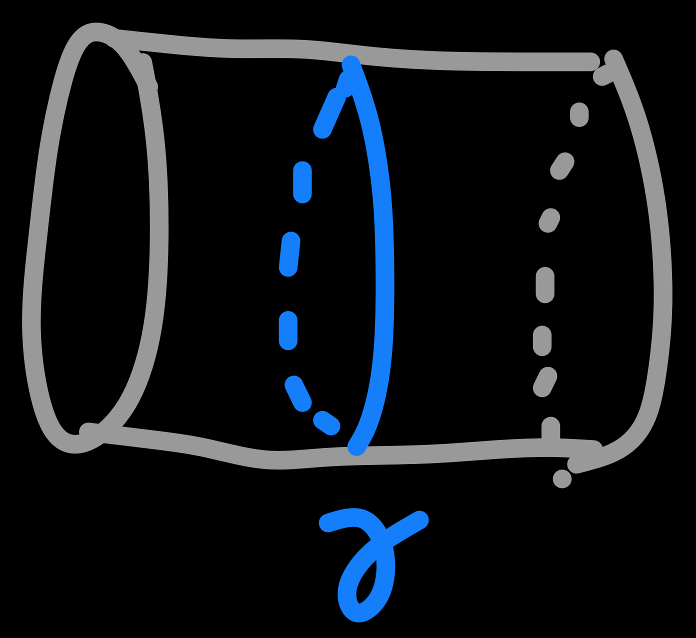

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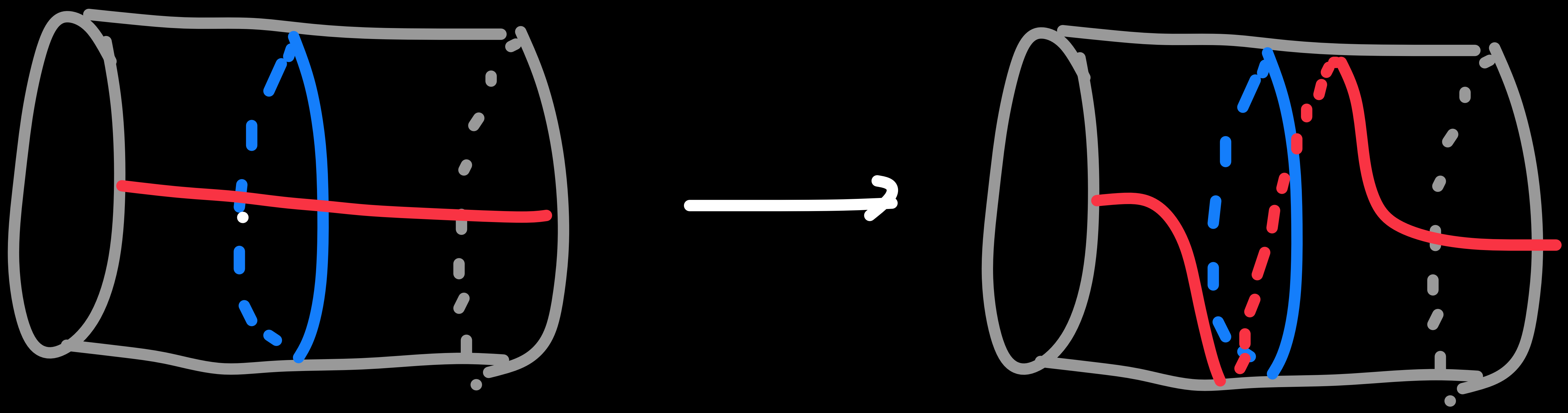

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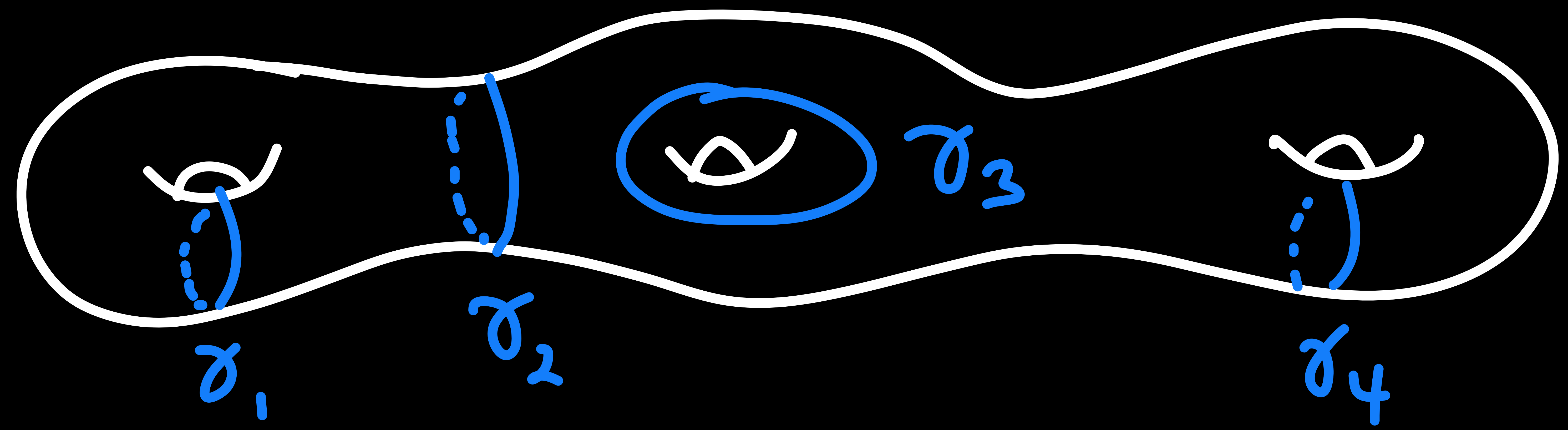

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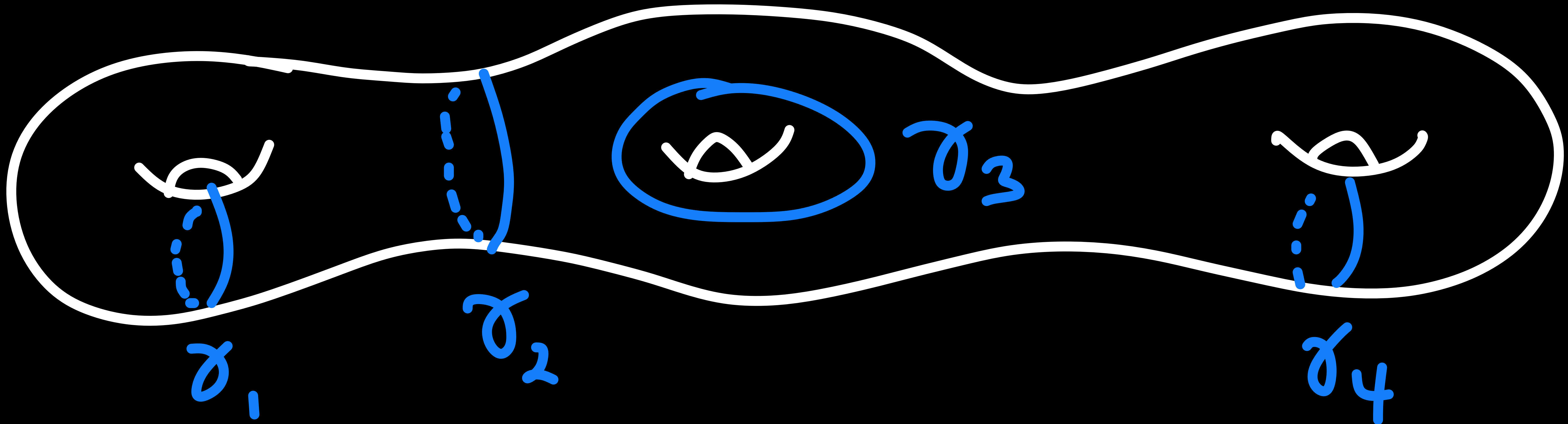
We call an element of the subgroup $\langle T_{\gamma_1}, \ldots, T_{\gamma_n} \rangle \cong \mathbb{Z}^n$ a *multitwist*.



Thank you!