Additive combinatorics from Combinatorial perspective

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Additive combinatorics

Setup:

- Let G be an abelian group.
- Given a finite $A \subseteq G$, define the sumset $A + A = \{a + b : a, b \in A\}$.
- Define the doubling $K = \frac{|A+A|}{|A|}$.

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Theme in additive combinatorics:

$$\text{Small doubling } K \qquad \leftrightarrow \qquad \text{Additively structured } A.$$

Additive structures and Freiman's theorem

Example:

- $G = \mathbb{Z}$.
- For any finite $A \subseteq \mathbb{Z}$:
  - $|A + A| \leq \binom{|A|+1}{2}$.
  - $|A + A| \geq 2|A| - 1$, with equality iff A is an arithmetic progression.

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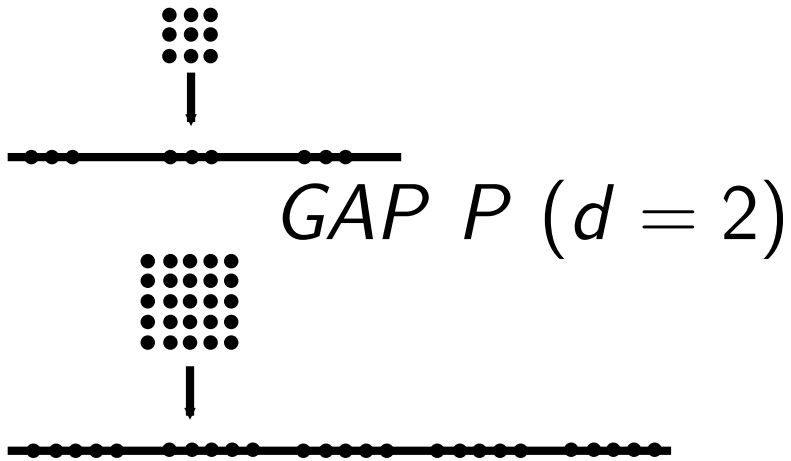
In general, for bounded d, a generalized arithmetic progression (GAP)

$$P := x_0 + \left\{ \sum_{i=1}^{d} a_i x_i : \ell_i \leq a_i \leq u_i \right\}$$

has a small doubling.



Additive structures and Freiman-Ruzsa's theorem

For $A \subseteq \mathbb{Z}$:

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In general, any dense subset of a generalized arithmetic progression with bounded dimension d has bounded doubling K.

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Theorem (Freiman's theorem '64, Ruzsa '92, '94)

If $A \subseteq \mathbb{Z}$ is so that $|A+A| \leq K|A|$, then A is contained in a GAP P of dimension $d = O_K(1)$ and size $|P| = O_K(|A|)$.

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Quantitative aspects of Freiman's theorem are of fundamental interest in additive combinatorics.

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Major theme in additive combinatorics:

A has small doubling $\frac{|A+A|}{|A|} \leq K \quad \Rightarrow \quad A$ is structured/ dense in a structured object.

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Drawback: Weak quantitative dependence on K, only applicable when K is very small compared to $|A|$.

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Many applications require to allow for K to grow in $|A|$ and motivate different notions of structure:

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- Combinatorial: "Forget" the group structure and move to general graph-theoretic representation of sets with small doubling.
- Probabilistic: Probe and approximate the structure of sets with small doubling via randomness.

New perspectives

New perspectives and new ways to quantify the complexity of sets with small doubling:

- Resolve old questions about classical notion of structure.
- Quantitatively efficient or nearly optimal.
- Provide nontrivial information already when $K = o(|A|)$.
- Combinatorial/Probabilistic perspective: Flexible and generalize significantly beyond additive setting.

New perspectives

Complexity notion Key ingredient	Applications
Expanding structures in sets with small doubling Main combinatorial lemma	**Ruzsa's conjecture;** **Counting sets with small doubling in general groups;** **Ramsey properties of random Cayley graphs - Alon's conjecture;** Robust Freiman-Ruzsa lemma; Random sumset extractors; Dimension of sets with small doubling
Low-complexity subsets of sumsets Efficient covering lemma	**Independence number of sparse random Cayley graphs;** Large sets which are not sumsets (Green); Structured subsets of sumsets of dense sets (Lovett)
Low-complexity approximations of sumsets Approximation lemma	**Sharp counting of sets with small doubling in abelian groups (Alon-Balogh-Morris-Samotij)**

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Question

What can we say about sets $A \subseteq G$ with $|A + A| \leq K|A|$?

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Theorem (Ruzsa's theorem, '99)

If G is an abelian group with exponent r and $A \subseteq G$ is so that $|A+A| \leq K|A|$, then A is contained in subgroup H of size $|H| = O_{r,K}(|A|)$.

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There is a constant $C > 0$ such that the following holds. If G is an abelian group with exponent r and $A \subseteq G$ is so that $|A+A| \leq K|A|$, then A is contained in subgroup H of size $|H| \leq r^{CK}|A|$.

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Example:

- $G = \mathbb{Z}_r^d$, $A = \{0, e_1, \ldots, e_d\}$.
- $|A+A| = \frac{(d+1)(d+2)}{2} = \frac{d+2}{2}|A|$.
- $|\langle A \rangle| = |G| = r^d$.

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Theorem (Gowers-Green-Manners-Tao, '24)

If G is an abelian group with exponent r and $A \subseteq G$ is so that $|A+A| \leq K|A|$, then there is a subgroup H with $|H| \leq K^C|A|$ for which A is covered by K^C translates of H.

Previous results

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Theorem (Sanders '12)

We can find $H \supseteq A$ with $|H| \leq r^{K(\log K)^{O(1)}}|A|$.

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For $G = \mathbb{F}_2^d$, we can find $H \supseteq A$ with $|H| \leq 2^{2K+O(\sqrt{K}\log K)}|A|$.

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Theorem (Even-Zohar – Lovett '14)

For $G = \mathbb{F}_p^d$, we can find $H \supseteq A$ with $|H| \leq \frac{p^{2K-2}}{2K-1}|A|$.

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There are cases in which groups with exponent divisible by more than one prime behave significantly different from those with prime power torsion!

Main resuilts

Theorem (Fox-P. '25+)

If G is an abelian group with exponent r and $A \subseteq G$ is so that $|A+A| \leq K|A|$, then A is contained in subgroup H of size $|H| \leq r^{(2+o(1))K}|A|$.

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Our key ingredient is the main combinatorial lemma producing *expanding structures* inside sets with small doubling.

The combinatorial setup & The key combinatorial lemma

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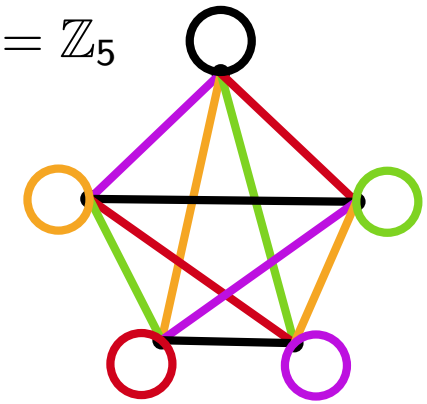

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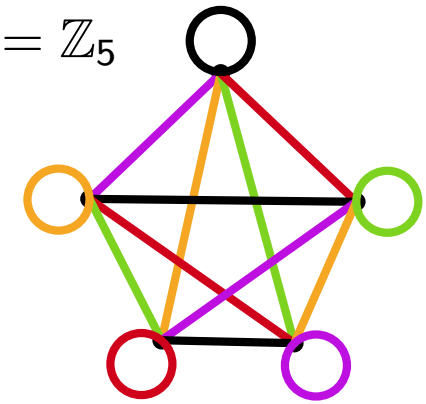

Surprise: Suffice to work in general combinatorial setup.

- Complete graph (on A) with a proper edge coloring using at most $K|A|$ many colors.

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Consider a proper edge coloring on A of size n using at most Kn colors, each appearing $O(n/K)$ times. There exists a set of $O(K)$ colors S such that the edges with colors in S partition A into sets B_i satisfying:

- At least $|A|/2$ vertices of A are contained in the sets B_i.
- Each B_i is connected using only edges with colors in S.
- For any set B_i and any $A' \subseteq A$ with $|A'| = \Omega(n)$, $N(B_i, A') = \Omega(Kn)$.

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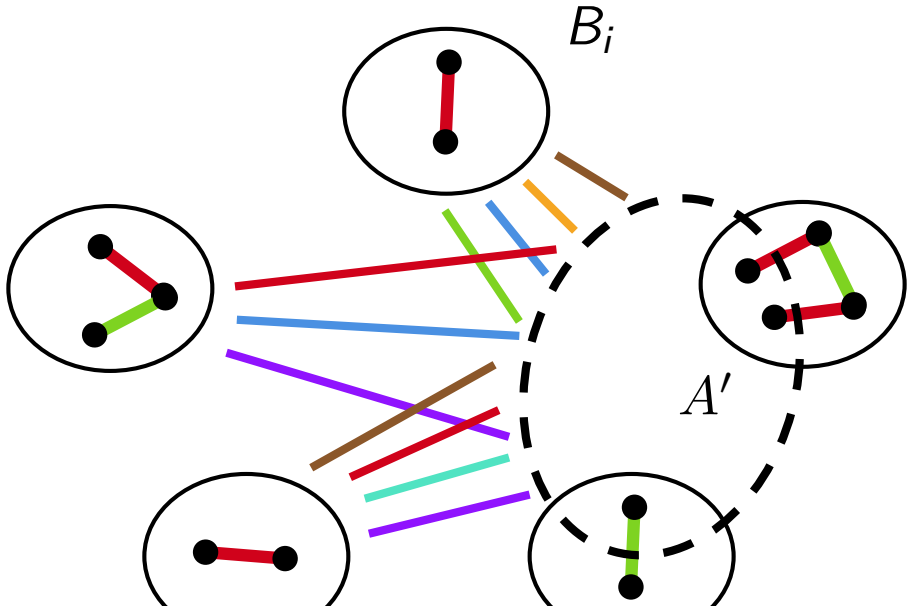
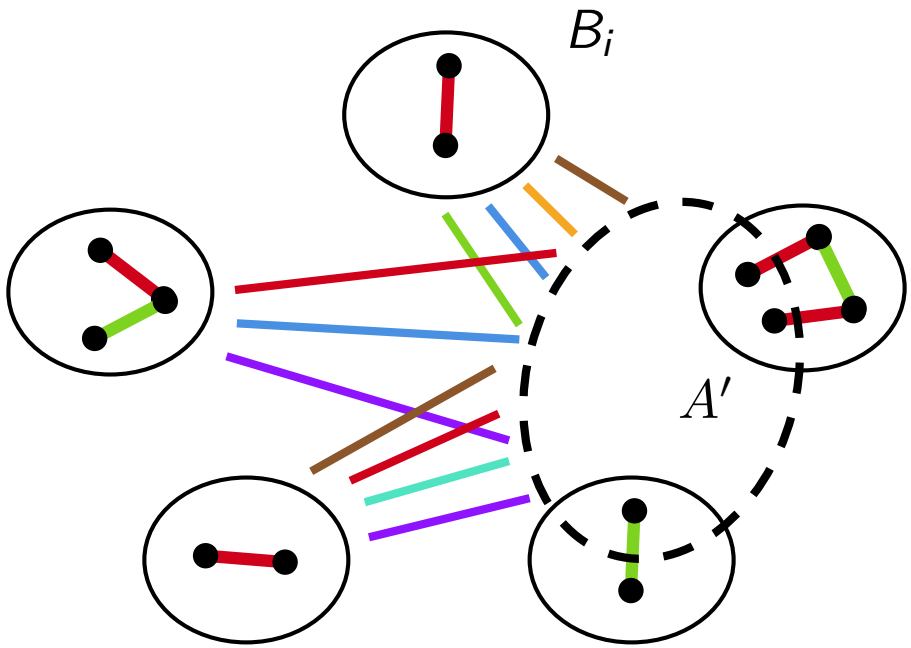


Approach to Ruzsa's conjecture

Step 1. Apply the key combinatorial lemma.



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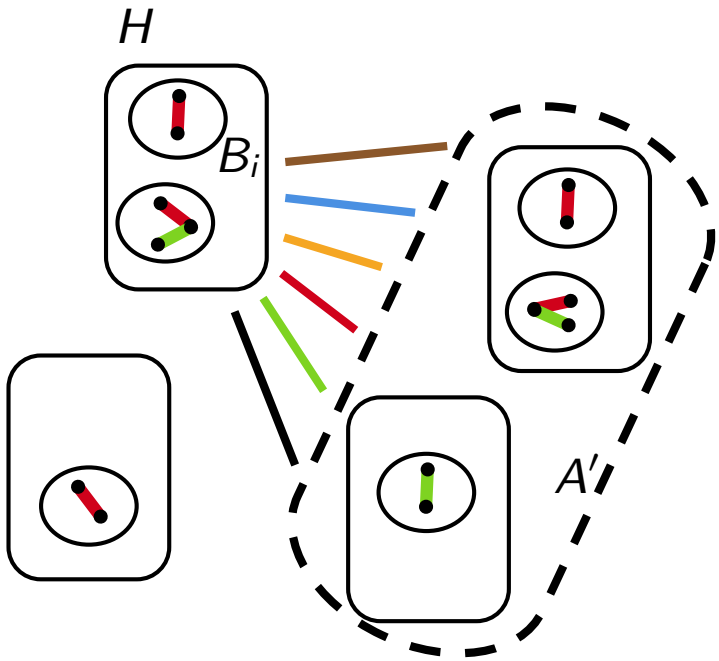

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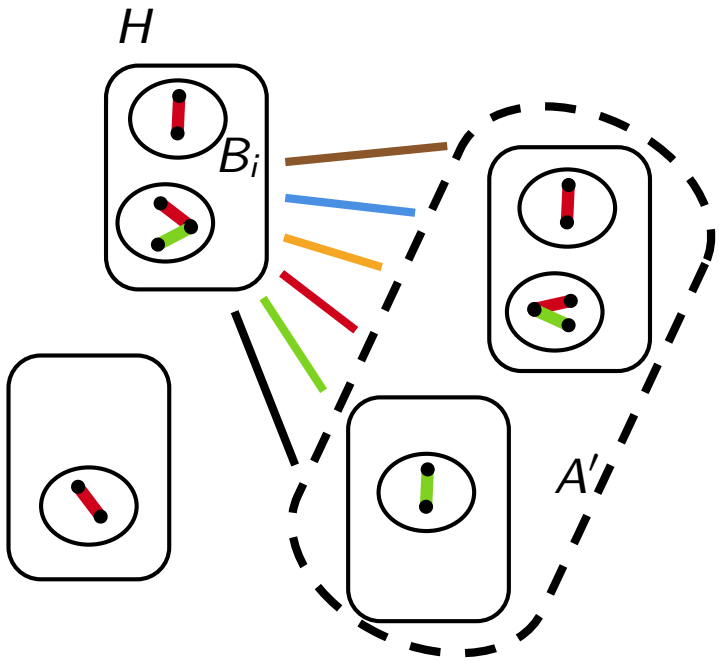

Step 3. Via qualitative version of Ruzsa's theorem, deduce that A is contained in a subgroup H with $|H| \leq |A| \exp(O(K))$.

Ramsey Cayley graphs, Dense Random Cayley graphs & Sets with small doubling in general groups

Ramsey graphs

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Theorem (Campos-Griffiths-Morris-Sahasrabudhe '23)

There is no $(\frac{1}{2} + c)$-Ramsey graph.

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There is no $(\frac{1}{2} + c)$-Ramsey graph.

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Almost all graphs on N vertices are 2-Ramsey.

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- One of the first applications of the probabilistic method.
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Problem (Erdős '47)

Explicitly construct C-Ramsey graphs for some constant C.

Ramsey Cayley graphs

Definition (Cayley graph)

For a group G and symmetric subset $S \subset G$, the *Cayley graph* G_S has vertex set G and distinct x, y are adjacent if $xy^{-1} \in S$.

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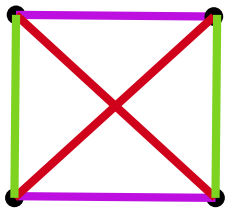

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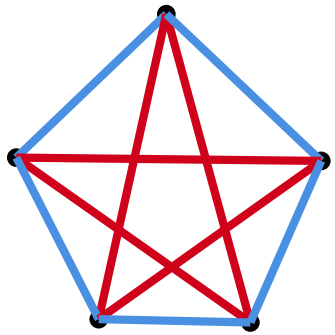

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Motivations:

- Small Ramsey graphs are Cayley; random analog of Paley graphs.
- Extensively studied in applications in theoretical computer science, combinatorics, number theory, group theory.
- Strong connections to coding theory, spectral graph theory, etc.

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There is a constant C such that every finite group has a Cayley graph which is C-Ramsey.

Random graphs meet additive combinatorics

Connection to additive combinatorics and group theory:

- For $A \subseteq G$, define the product set

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In an abelian group, this is the difference set $A - A$.

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The first moment of the number of independent sets in a random Cayley graph is intimately related to the number of sets with small product sets:

$$\mathbb{E}[\#\text{independent sets of size } t \text{ in } G(p)] = \sum_{|A|=t} (1-p)^{\Theta(|AA^{-1}|)}.$$

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Controlling the first moment of the number of independent sets in a random Cayley graph reduces to bounding the number of sets with small product sets.

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Theorem (Conlon-Fox-P.-Yepremyan '24)

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Our proof of the theorem is entirely combinatorial!

Combinatorial view on Cayley graphs

Combinatorial view of the group structure:

- Complete graph on G with an edge coloring $c(\{x, y\}) = \{xy^{-1}, yx^{-1}\}$.
  - In this edge-coloring each color class is 1 or 2-regular.

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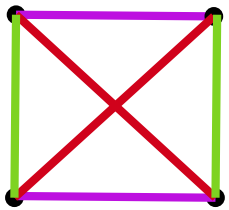

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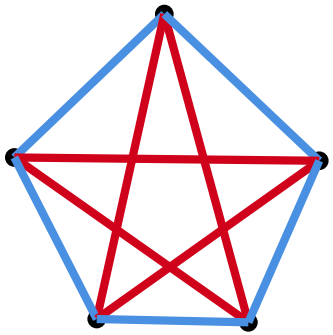

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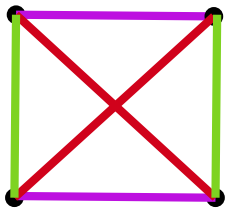

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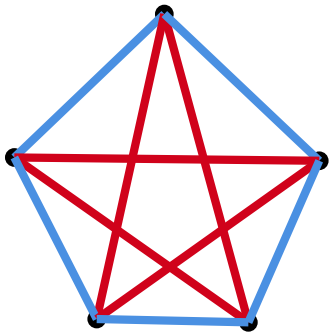

Surprise: The combinatorial constraint on the degree of color classes is sufficient!

Counting sets with small product set

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In a Δ-bounded edge-coloring of the complete graph on N vertices, the number of n-vertex subsets with at most Kn colors is at most

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The lemma follows from the key combinatorial lemma.

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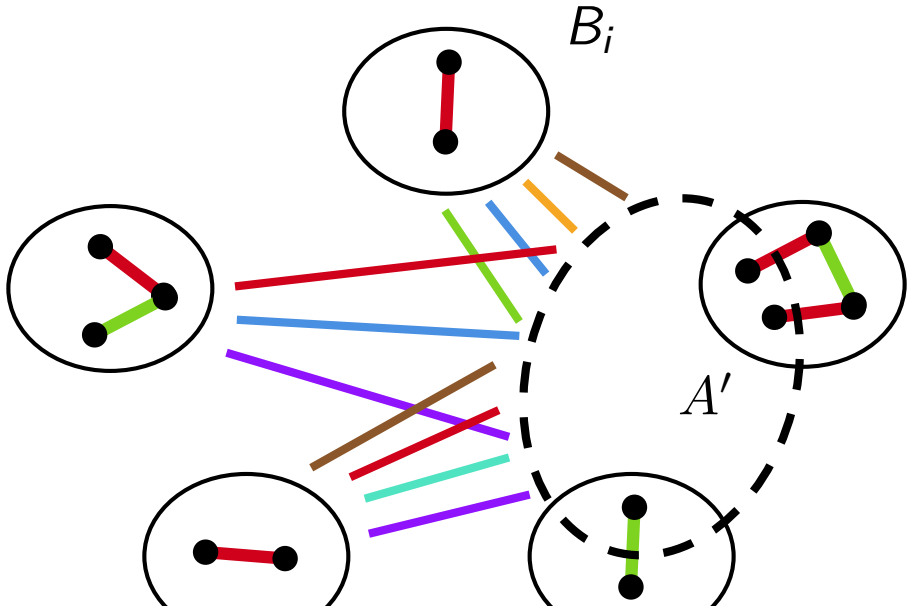
Consider a proper edge coloring on A of size n using at most Kn colors, each appearing $O(n/K)$ times. There exists a set of $O(K)$ colors S such that the edges with colors in S partition A into sets B_i satisfying:

- At least $|A|/2$ vertices of A are contained in the sets B_i.
- Each B_i is connected using only edges with colors in S.
- For any set B_i and any $A' \subseteq A$ with $|A'| = \Omega(n)$, $N(B_i, A') = \Omega(Kn)$.



Alon's conjecture - Going beyond uniform random

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For $p \geq 5$, there exists Cayley graphs over $\mathbb{F}_p^d$ with clique and independence number $(2 + o(1)) \log_2 N$ where $N = p^d$.
For $p = 1 \pmod 4$, these Cayley graphs are self-complementary.

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Recent generalization to all groups of order coprime to 6 by Schildkraut.

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The first moment over those A, together with the previous counting result, imply the Ramsey property.

Perspective

Cliques in dense random Cayley graphs $\leftrightarrow$ Counting sets with small doubling.

Combinatorial perspective: Main combinatorial lemma identifies novel combinatorial structures underlying sets with small doubling.

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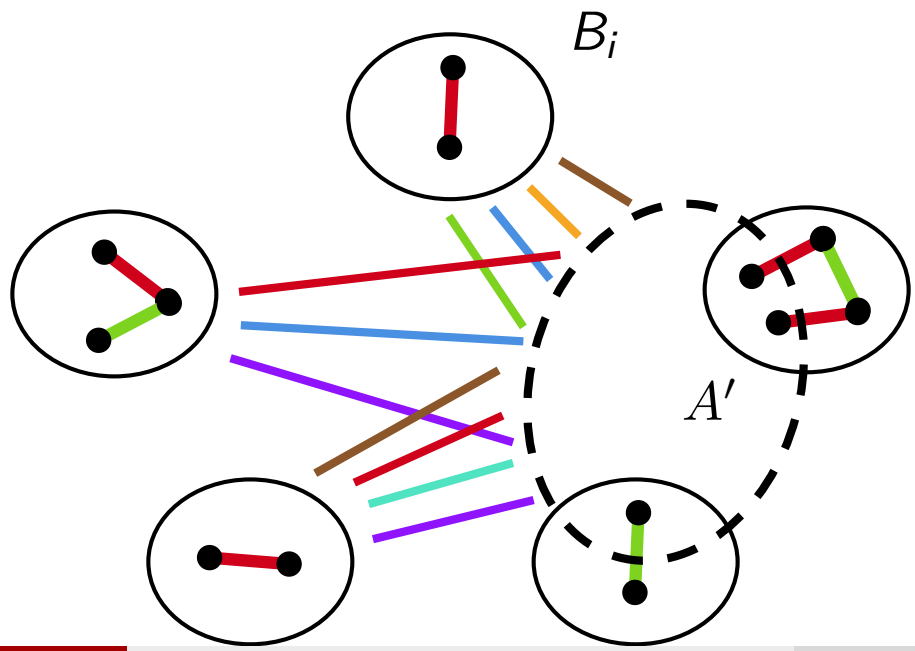
Combinatorial perspective: Main combinatorial lemma identifies novel combinatorial structures underlying sets with small doubling.

New combinatorial approach to additive combinatorics:

- Main combinatorial lemma: expanding structures in sets with small doubling.
- Refinement: Use expanding structures to probe information about the entire set.



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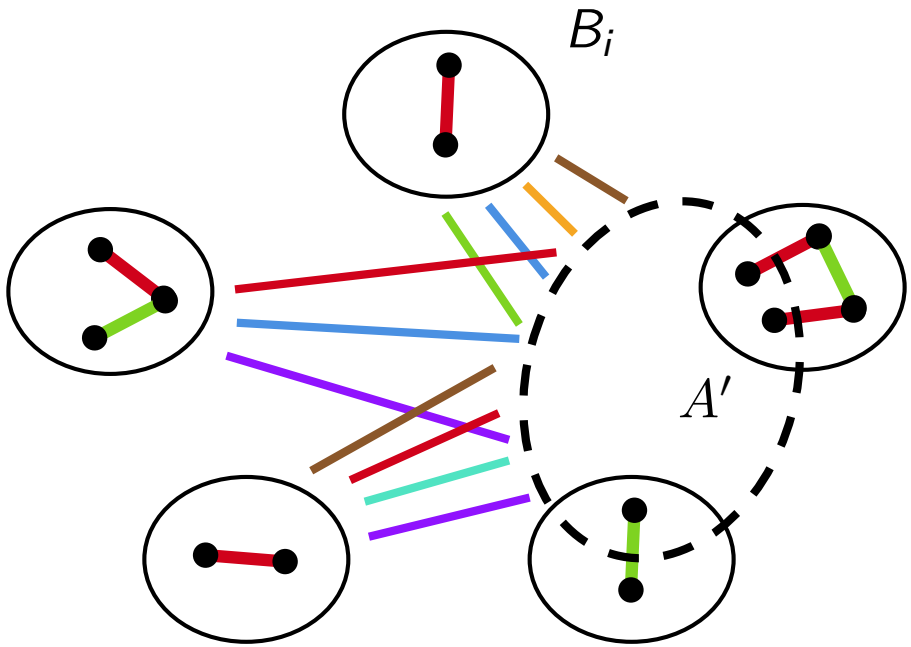

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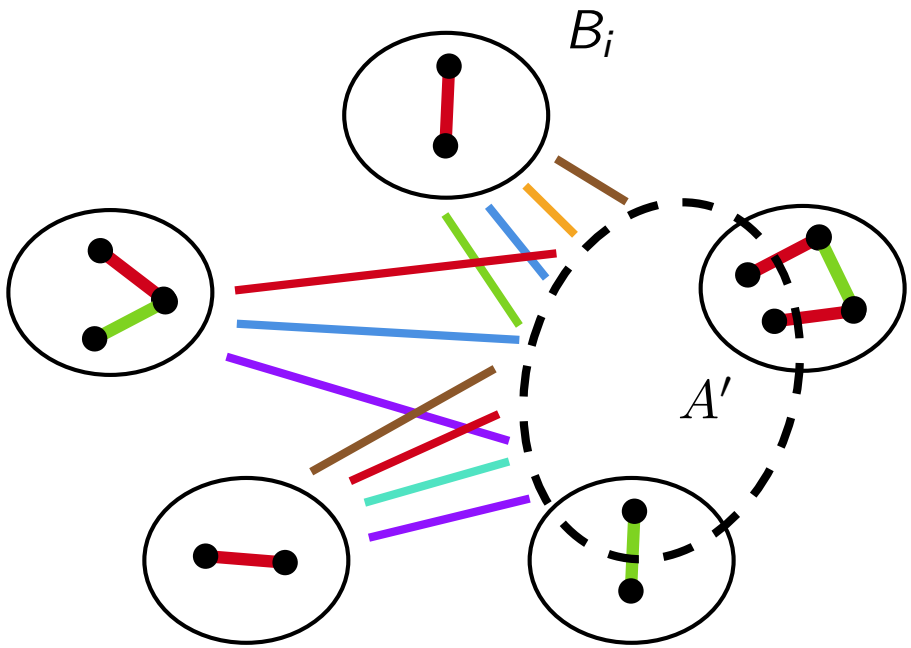

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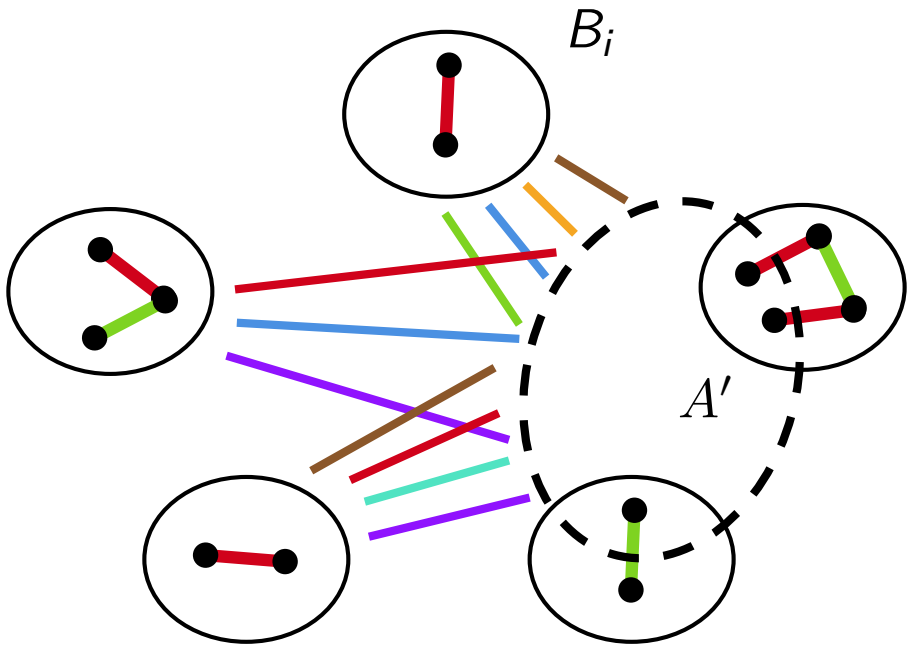

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Robust and applicable in wide generality.

New direction: Dependent random graphs

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Examples:

- Erdős-Rényi random graphs.
- Random Cayley graphs.
- Random Latin square graphs: Color class $C_k = \{\{i,j\} : L_{ij} = k\}$ for a Latin square L.

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- From Theorem 1, a careful union bound yields Theorem 2.
- Theorem 2 solves a conjecture of Christofides and Markström ('11) on the independence number of random Latin square graphs.

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How much independence is needed?

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The phenomena extend to a significantly broader class of random graphs.

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An ensemble of random graphs is said to be Δ-independent if for each edge e, there is a graph G_e of maximum degree Δ such that the appearance of e is independent of all edges outside G_e.

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Theorem (Conlon-Fox-P.-Yepremyan '26+)

Consider a Δ-independent random graph G where the probability of appearance of each edge is $\Theta(p)$. Then, with high probability,

$$\alpha(G) = O_{p,\Delta}(\log N \log\log N).$$

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Consider a symmetric Δ-independent random graph G where the probability of appearance of each edge is p. Then, with high probability, all nontrivial eigenvalues of G are bounded by $O(\sqrt{pN \log N})$.

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Open direction

Study interesting properties of Δ-independent graphs (random entangled graphs, random Cayley graphs).

The next part

Probabilistic approach to sets with small doubling:

- Study independent sets in sparse random Cayley graphs.

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- Study independent sets in sparse random Cayley graphs.
- Via the connection between threshold phenomena and first moment obstructions, probabilistic predictions **suggest** existence of significant low-complexity structures among sets with small doubling.
- New approach to probe low-complexity structures:
  - Progress on understanding sparse random Cayley graphs.
  - Much finer understanding of sets with small doubling, Optimal enumeration results.

Thank you!