Groupoids in Operator Algebra and Abstract Algebra: Part 2

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Becky Armstrong

VICTORIA UNIVERSITY OF WELLINGTON
TE HERENGA WAKA

Groupoidology

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Examples: Groups, equivalence relations, group actions, directed graph groupoids, etc.

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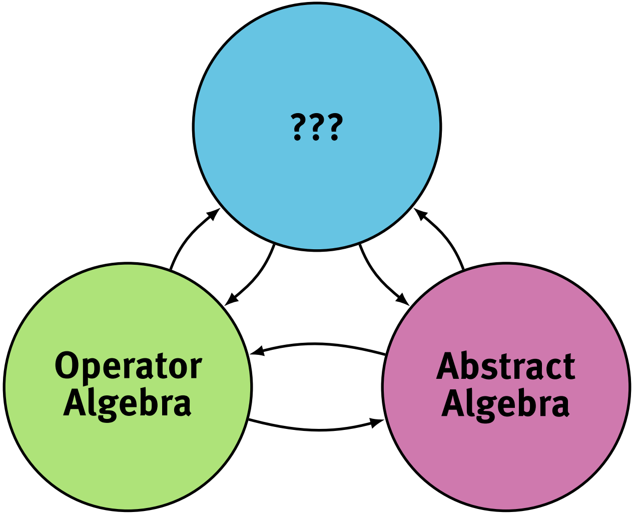
We say that $\mathcal{G}$ is **ample** if it has a **basis of *compact* open bisections** (called **"cobs"**). Given any ample Hausdorff groupoid $\mathcal{G}$ and commutative unital ring R, there is an associated **Steinberg algebra** $A_R(\mathcal{G})$.

My research



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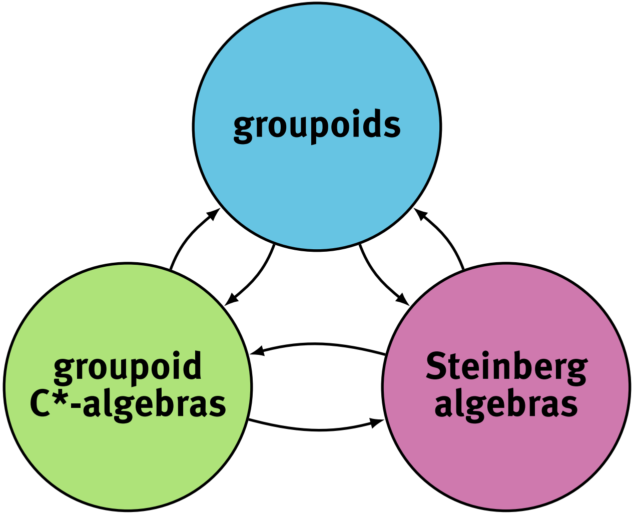
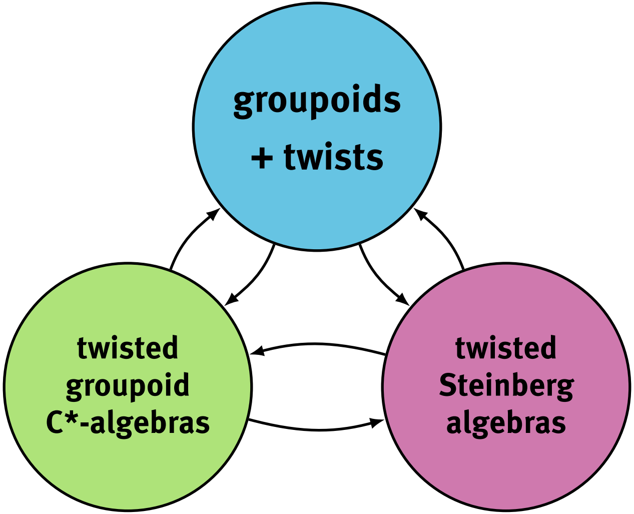


My research: twisted groupoid algebras



Structure Theory

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Question: How can we tell whether a given groupoid C*-algebra or Steinberg algebra is simple?

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(b) *If $\mathcal{G}$ is second-countable and amenable, then $C^*(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and effective.*

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Then $C_c(\mathcal{G}, \sigma)$ is a $*$-algebra. We complete $C_c(\mathcal{G}, \sigma)$ with respect to full and reduced norms defined analogously to the non-twisted setting to obtain the **full** and **reduced twisted groupoid C*-algebras** $C^*(\mathcal{G}, \sigma)$ and $C_r^*(\mathcal{G}, \sigma)$, respectively.

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Open question: What happens if $\mathcal{G}$ is minimal and not effective?

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So we need a different simplicity characterisation for twisted groupoid C*-algebras.

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The topological interior $\mathcal{I}_T$ of $\text{Iso}(\mathcal{G}_T)$ is an amenable Hausdorff étale groupoid.

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We call P_T the **periodicity group** of $\mathcal{G}_T$.

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Lemma (A–Brownlowe–Sims 2024)

Every cohomology class of a minimal Deaconu–Renault groupoid $\mathcal{G}_T$ contains a continuous 2-cocycle σ such that $\sigma|_{\mathcal{I}_T^{(2)}} = 1_X \times \omega$ for some bicharacter ω of P_T that vanishes on Z_ω.

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There is an action $\theta\colon \mathcal{G}_T/\mathcal{I}_T \curvearrowright X \times \widehat{Z_\omega}$ that captures all the twisting that occurs when conjugating a function in $C_c(\mathcal{I}_T, \sigma)$ by one in $C_c(\mathcal{G}_T, \sigma)$.

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Future work: Characterise simplicity of twisted Steinberg algebras of ample Deaconu–Renault groupoids.

Reconstruction Theory

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for $(\alpha, \beta) \in \mathcal{G}^{(2)}$ and $w, z \in \mathbb{T}$. In fact, $\mathcal{E}_\sigma$ is a twist by $\mathbb{T}$ over $\mathcal{G}$.

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2-cocycles on discrete groups are in one-to-one correspondence with group extensions. A **twist** (or **twisted groupoid**) $\mathcal{E}$ by $\mathbb{T}$ over a Hausdorff étale groupoid $\mathcal{G}$ is the groupoid analogue of a group extension.

Example

Let $\mathcal{G}$ be a Hausdorff étale groupoid, and let $\sigma\colon \mathcal{G}^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle. Give $\mathcal{E}_\sigma \coloneqq \mathcal{G} \times \mathbb{T}$ the product topology. Then $\mathcal{E}_\sigma$ is a Hausdorff groupoid under the operations

$$(\alpha, w)(\beta, z) \coloneqq \big(\alpha\beta,\ \sigma(\alpha, \beta)\, wz\big) \qquad \text{and} \qquad (\alpha, w)^{-1} \coloneqq \big(\alpha^{-1},\ \overline{\sigma(\alpha, \alpha^{-1})}\, \overline{w}\big)$$

for $(\alpha, \beta) \in \mathcal{G}^{(2)}$ and $w, z \in \mathbb{T}$. In fact, $\mathcal{E}_\sigma$ is a twist by $\mathbb{T}$ over $\mathcal{G}$.

Unlike for groups, not every twisted groupoid is induced by a continuous 2-cocycle [Kumjian 1986, ANSZ 2025].

Twisted groupoid C*-algebras

Given a twist $\mathcal{E}$ by $\mathbb{T}$ over a Hausdorff étale groupoid $\mathcal{G}$, we can construct **full** and **reduced twisted groupoid C*-algebras** $C^*(\mathcal{G};\mathcal{E})$ and $C^*_r(\mathcal{G};\mathcal{E})$.

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If the twist $\mathcal{E}$ is induced by a continuous 2-cocycle $\sigma\colon \mathcal{G}^{(2)} \to \mathbb{T}$, then these C*-algebras coincide with $C^*(\mathcal{G}, \sigma)$ and $C_r^*(\mathcal{G}, \sigma)$, respectively.

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In [ACCCLMRSS 2023], my coauthors and I introduced the notion of a **discrete twist** $\mathcal{E}$ by $R^\times$ over $\mathcal{G}$, that doesn't necessarily arise from a continuous $R^\times$-valued 2-cocycle on $\mathcal{G}$.

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We then defined an associated **twisted Steinberg algebra** $A_R(\mathcal{G}; \mathcal{E})$, generalising those defined using continuous 2-cocycles.

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Theorem (Li 2020)

Every classifiable C-algebra has a Cartan subalgebra.*

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Theorem (Kumjian 1986, Renault 2008)

If B is a Cartan subalgebra of a C-algebra A, then there is a unique twisted groupoid $(\mathcal{G}, \mathcal{E})$ and an isomorphism $\Psi\colon A \to C_r^*(\mathcal{G}; \mathcal{E})$ such that $\Psi(B) = C_0(\mathcal{G}^{(0)})$.*

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Open question: Is *every* C*-algebra a twisted groupoid C*-algebra?

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Current work: Prove existence and uniqueness theorems for abstract-algebraic Cartan pairs, and extend the theory to cover R-rings rather than just R-algebras.

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Thanks!