Additive combinatorics: Probabilistic perspective

Huy Tuan Pham

California Institute of Technology

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Recap

Definition (Cayley graph)

For a group G and symmetric subset $S \subset G$, the *Cayley graph* G_S has vertex set G and distinct x, y are adjacent if $xy^{-1} \in S$.

Given $p \in (0,1)$, the random Cayley graphs $G(p)$ is the Cayley graph G_S where each $\{g, g^{-1}\}$ is included independently in S with probability p.

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With high probability, the independence and clique number of a uniform random Cayley graph on any group G of order N is $O(\log N \log\log N)$.

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If G is an abelian group with exponent r and $A \subseteq G$ is so that $|A+A| \leq K|A|$, then A is contained in subgroup H of size $|H| \leq r^{(2+o(1))K}|A|$.

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Further applications:

- Dimension of sets with small doubling
- Robust Freiman-Ruzsa lemma
- Random sumset extractors

Perspective

Combinatorial perspective: Main combinatorial lemma identifies novel combinatorial structures underlying sets with small doubling.

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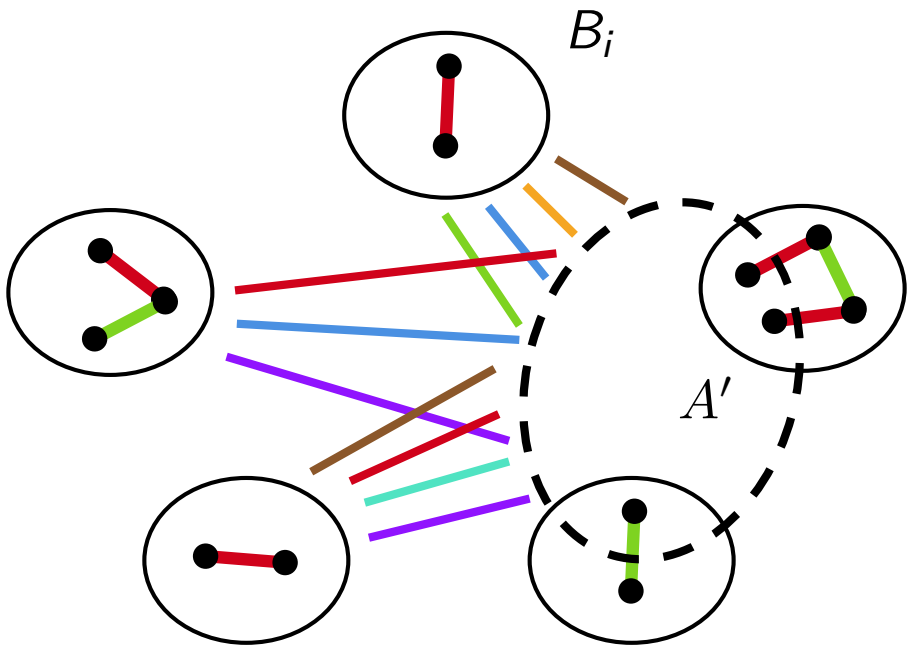
New combinatorial approach to additive combinatorics:

- Main combinatorial lemma: expanding structures in sets with small doubling.
- Refinement: Use expanding structures to probe information about the entire set.



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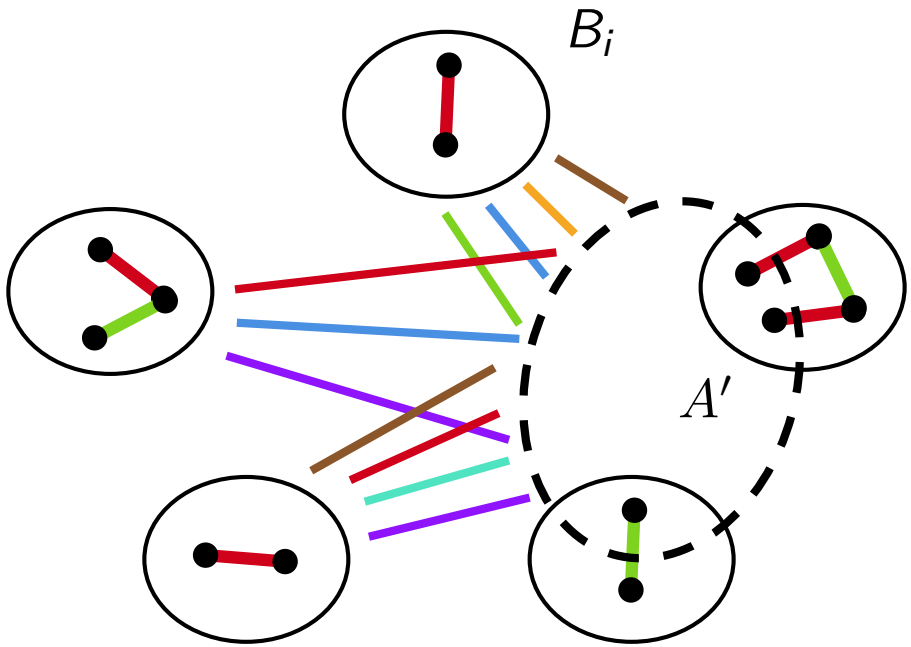

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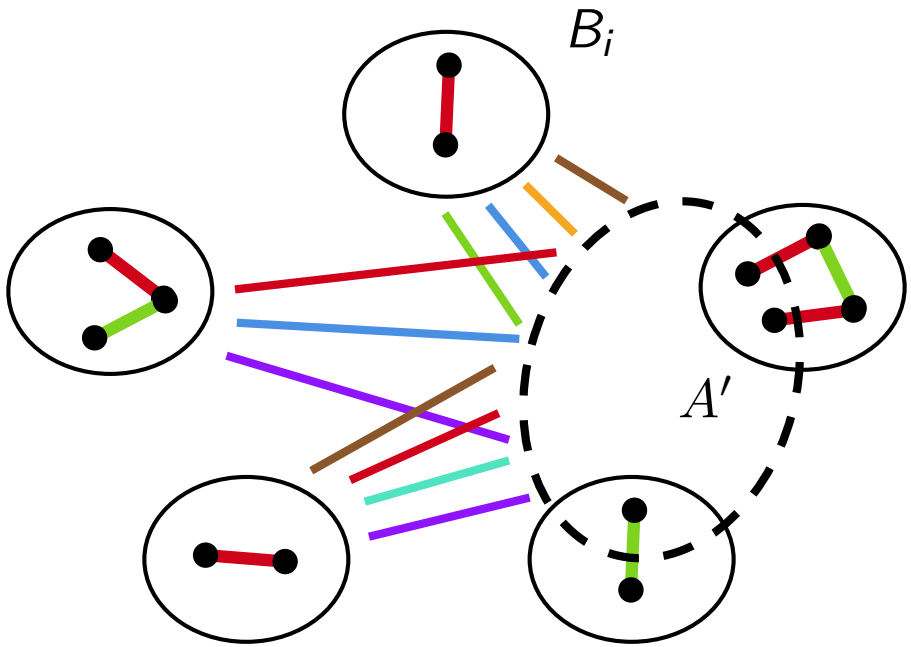

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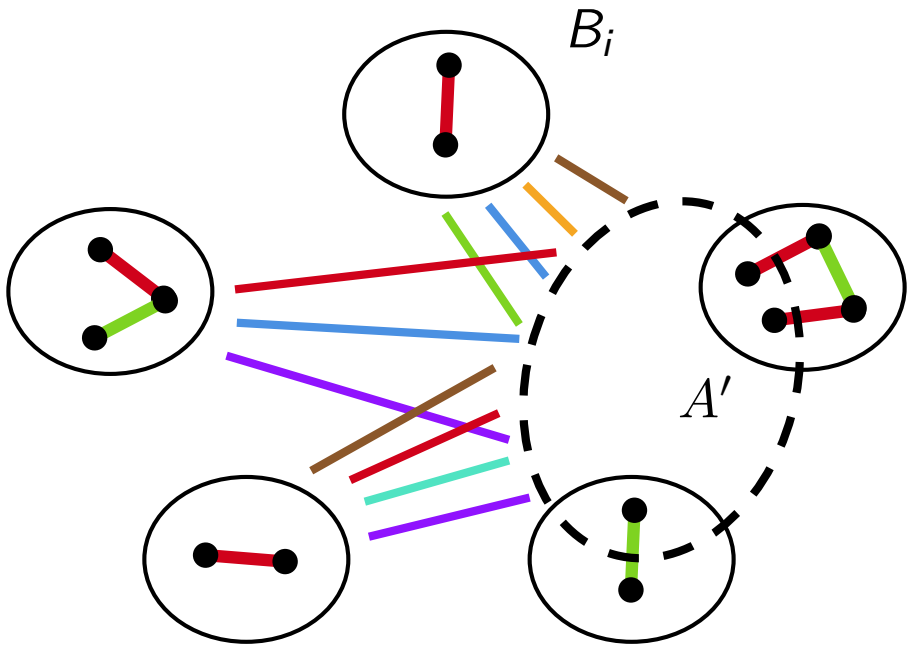

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Robust and applicable in wide generality.

Alon's conjecture - Going beyond uniform random

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Theorem (Conlon-Fox-P.-Yepremyan '24)

For $p \geq 5$, there exists Cayley graphs over $\mathbb{F}_p^d$ with clique and independence number $(2 + o(1)) \log_2 N$ where $N = p^d$.
For $p = 1 \pmod 4$, these Cayley graphs are self-complementary.

Answer a question of Alon and Orlitsky ('95) motivated by zero-error capacity and dual-source coding.

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Recent generalization to all groups of order coprime to 6 by Schildkraut.

Dependent random graphs

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Examples:

- Erdős-Rényi random graphs.
- Random Cayley graphs.
- Random Latin square graphs: Color class $C_k = \{\{i,j\} : L_{ij} = k\}$ for a Latin square L.

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Theorem 1 (Conlon-Fox-P.-Yepremyan '24)

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Theorem 2 (Conlon-Fox-P.-Yepremyan '24)

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- From Theorem 1, a careful union bound yields Theorem 2.
- Theorem 2 solves a conjecture of Christofides and Markström ('11) on the independence number of random Latin square graphs.

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Theorem (Conlon-Fox-P.-Yepremyan '26+)

Consider a Δ-independent random graph G where the probability of appearance of each edge is $\Theta(p)$. Then, with high probability,

$$\alpha(G) = O_{p,\Delta}(\log N \log\log N).$$

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Open direction

Study interesting properties of Δ-independent graphs (random entangled graphs, random Cayley graphs).

Sparse random Cayley graphs & First moment obstructions

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Conjecture (Alon '07, '13)

The independence number of $G(p)$ is with high probability $\tilde{O}(p^{-1})$.

The random Cayley sum graph $G(p)$ behave like a random pN-regular graph.

First moment obstructions and Additive combinatorics

Naive first moment for independent sets in Cayley sum graphs:

$$\mathbb{E}[\#\text{ independent sets of size } t \text{ in } G(p)] = \sum_{|A|=t} (1-p)^{|A+A|} \approx \sum_{|A|=t} \exp(-p|A+A|).$$

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Need better First moment obstructions.

First moment obstructions

General setup:

- Finite set X.
- Collection of target sets $\mathcal{H} \subseteq 2^X$.
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Denote $\langle \mathcal{H} \rangle$ the collection of subsets $W \in 2^X$ containing at least one target set $H \in \mathcal{H}$:

Question

Does X_p contain a target set $H \in \mathcal{H}$?

What are obstructions (certificates) which imply that $\mathbb{P}(X_p \in \langle \mathcal{H} \rangle)$ is small?

First moment obstructions

Example:

- $X = \binom{[N]}{2}$, then $X_p \sim G(N, p)$.
- $\mathcal{H}$ is the collection of isomorphic copies of a graph H on N vertices.
  - In Erdős' case, H is taken to be a clique on $n = 2\log_2 N$ vertices.

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Example: Erdős' proof that random graphs $G(N,1/2)$ do not have cliques of size $2\log_2 N$.

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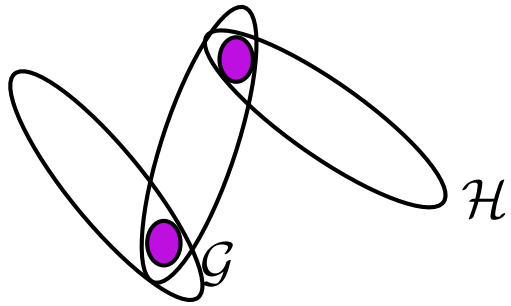
We say that $\mathcal{G}$ is a cover for $\mathcal{H}$ if every $H \in \mathcal{H}$ contains some $G \in \mathcal{G}$.



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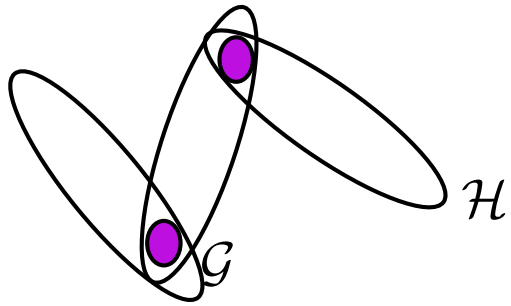

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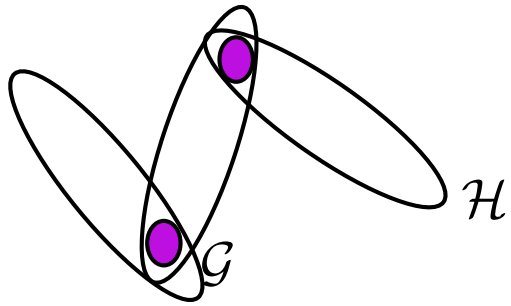

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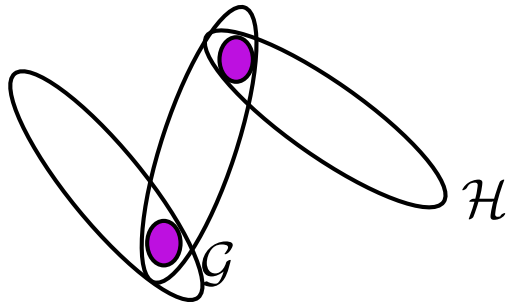

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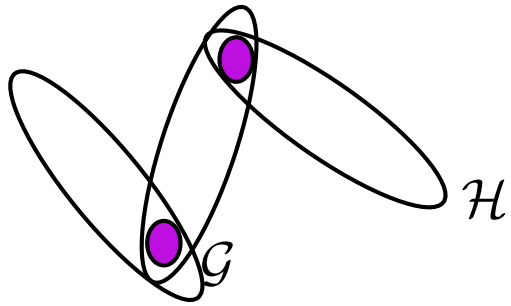

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If $\mathcal{H}$ is p-small, then $\mathbb{P}(X_p \in \langle \mathcal{H} \rangle) < 1/2$.

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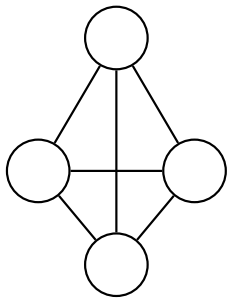

Figure 1: $H = K_4$

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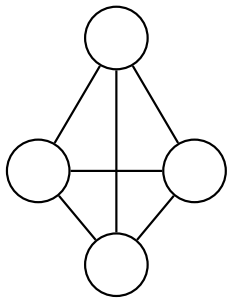

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- For $p < cn^{-2/3}$, there is a first moment obstruction showing that $\mathbb{P}(G(N,p) \text{ contains } K_4) < 1/2$.

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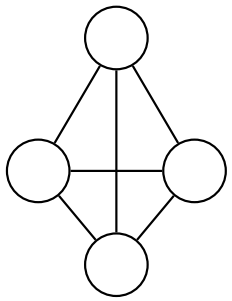

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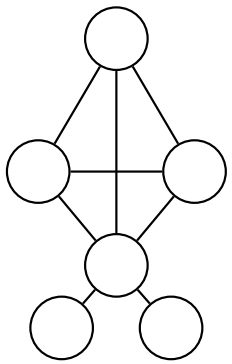

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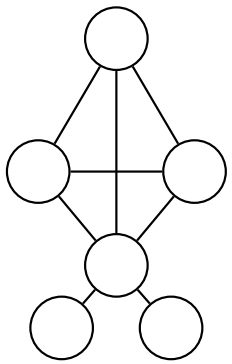

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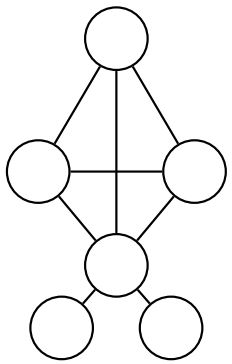

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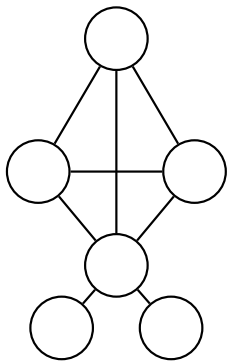

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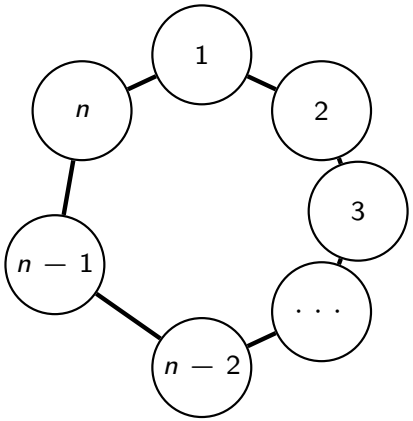

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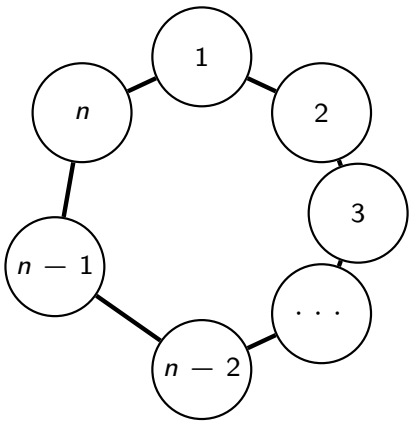

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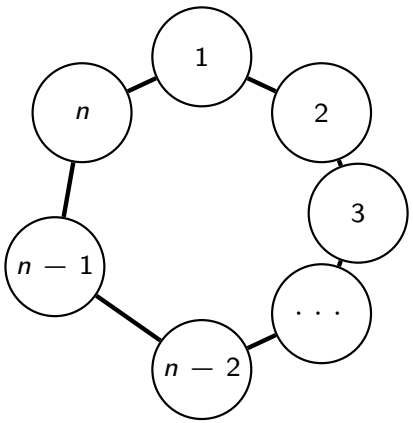

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- Other covers: all edges adjacent to a fixed vertex.
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Many other interesting manifestation of the central philosophy in probability, high-dimensional geometry, etc.

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New perspective:

First moment obstructions = Existence of large, low-complexity substructures.

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Perspective: New ways to quantify the structure of sets with small doubling.

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Define the complexity of a collection of sets $\mathcal{F}$ as $\log |\mathcal{F}|$.

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Fundamental barrier at $t^{1/2}$:

- For A with small doubling $K = \frac{|A+A|}{|A|}$, $|A + A| \leq |A|^{1+\delta}$, any improvement must leverage suitably additional structure.

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Among $|A| = t$ with small sumset $A + A$, does $A + A$ contain large low-complexity subsets?

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Beside weak quantitative bounds, even when A is dense in G, existence of low-complexity cover is open.

Question (Lovett)

For $G = \mathbb{F}_2^d$, does there exist a collection of dense subsets of G with complexity $d^{O(1)}$ which covers the collection of sumsets $A + A$ where $|A| = \Omega(2^d)$?

The main covering lemma

Efficient covering lemma (informal version)

There exists a collection of sets $\mathcal{C}$ such that:

- $|\mathcal{C}| \leq \exp(\tilde{O}(\min(K^2, \sqrt{Kt}))$.
- Every $C \in \mathcal{C}$ has $|C| \geq \tilde{\Omega}(Kt)$.
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- In the range K large, use a random sampling argument in physical space.

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Probabilistic approximation:

- The hard range (K small) performs approximation in the Fourier space:

$$A * A(x) = \sum_{\chi} \hat{A}(\chi)^2 \chi(x).$$

- Sample $\approx K^2$ random characters χ according to $|\hat{A}(\chi)|^2$, and construct a suitable Fourier-sparse pointwise approximation of $A * A$.

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Question

Can we achieve a low-complexity approximation with optimal dependence on K?

Optimally counting sets with small doubling

Optimal complexity for approximating sumsets

Enumeration of sets with small doubling

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Except for very small K ($K < 3$), structural method is too weak for meaningful enumeration.

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- In $\mathbb{F}_p^d$: subgroups of size $t \sim d \log d$.

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The examples show that the conjecture can only hold for $K \ll t/(\log N \log\log N)$.

Optimally counting sets with small doubling

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Consider an abelian group G of order N. Let $K \leq t/(\log N \log\log N)$. The number of $A \subseteq G$ with $|A| = t$ and $|A + A| = Kt$ is $\exp(o(t))\binom{(Kt+s)/2}{t}$, where s is the maximum size of a subgroup of size at most Kt.

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Progress:

- Green-Morris '16: $K = O(1)$.
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Technique:

- Hypergraph container lemma.
- Regularity-type method.

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For every t and K, there exists $\mathcal{F}$ with $|\mathcal{F}| \leq \exp(K(\log N)(\log\log N))$ such that: For every $|A| = t$ with $|A + A| = Kt$, there exists $F \in \mathcal{F}$ such that $|F \Delta (A + A)| \leq o(Kt)$.

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More accurately, the approximation applies to the subset B of $A + A$ consisting of elements x with at least $\epsilon t/K$ representations as $a_1 + a_2$.

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This complexity is optimal.

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By relaxing the one-sided covering condition, we can attain optimal complexity for approximating $A+A$.

This is sufficient for enumeration of sets with small doubling.

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Recover enumeration of A from Approximation lemma:

- Find F of low-complexity which approximates $A+A$.
- Based on F, construct a superset $X \supset A$.
- Via a graph container algorithm, show that we can efficiently refine until $|X| \leq (1+o(1))Kt/2$.

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The approximation lemma relies on a probabilistic approximation simultaneously in the Fourier and physical space:

- Based on a small random sample of a subset of A, construct an approximation $\hat{f}$ of $\hat{A}$.
- Smoothen $\hat{f}$, and apply Fourier inversion on $\hat{f}^2$.

Refined counting

Question (Green-Morris '16)

What is the number of $A \subseteq \mathbb{Z}_N$ with $|A| = t = \Theta(\log N)$ and $|A + A| \leq Kt$?

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Rely on approximation framework together with a crucial additional ingredient: Robust version of Freiman-Ruzsa's lemma over $\mathbb{Z}_N$.

- Established in Alon-P. '25, for sharp asymptotics of the independence number of logarithmically sparse random Cayley graph.
- Proof relies on the main combinatorial lemma.

Conclusion

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First moment obstructions suggest existence of low-complexity substructures.

Randomized approximations provide a pathway to Low-complexity approximations (and hence First moment obstructions).

Outlook: Low-complexity approximations

Our low-complexity covers and approximations provide explicit classes of structured functions that approximate large level sets of the Fourier transform.

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There exists a collection of sets of size $\tilde{\Omega}(Kt)$ of complexity $\tilde{O}(K)$ which cover $A+A$ where $|A| = t$ and $|A+A| \leq Kt$.

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Directions

- Further applications of low-complexity approximations beyond the additive combinatorial context.
- Alon's conjecture in groups with exponent 2 and 3.
- First moment obstructions beyond random graphs: Study interesting properties of Δ-independent graphs (random entangled graphs, random Cayley graphs).

Thank you!