Symmetries in noncommuting variables: in and **out**

Part Two

Jurij Volčič

University of Auckland

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The plan

- Emmy Noether's Problem
- Noncommutative polynomials and rational functions
- Free Noether & Lüroth Problems
- ToDo

Throughout: $\mathbb{k}$ is a ground field

A familiar example

let S_2 act on $\mathbb{k}^2$ by swapping the coordinates $x \leftrightarrow y$

invariant functions f on $\mathbb{k}^2$, such that $f(x, y) = f(y, x)$?

symmetric polynomials: $\mathbb{k}[x + y, xy]$
e.g. $x^2 + y^2 = (x + y)^2 - 2 \cdot xy$

symmetric rational functions: $\mathbb{k}(x + y, xy)$

Classical invariant theory

Let G be a group acting linearly on the vector space $\mathbb{k}^n$

G induces a linear action on $\mathbb{k}[x_1, \ldots, x_n]$ and $\mathbb{k}(x_1, \ldots, x_n)$

G-invariant polynomials: $\mathbb{k}[x_1, \ldots, x_n]^G$

G-invariant rational functions: $\mathbb{k}(x_1, \ldots, x_n)^G$

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Basic queries:

- What is a small set of generators of $\mathbb{k}[x_1, \ldots, x_n]^G$ as a $\mathbb{k}$-algebra?

  What is a small set of generators of $\mathbb{k}(x_1, \ldots, x_n)^G$ as a $\mathbb{k}$-field?

  1$^{\text{st}}$ fundamental theorem
- What are the relations between these generators?

  2$^{\text{nd}}$ fundamental theorem

Classical invariant theory

Let G be a group of $\mathbb{k}$-linear automorphisms of $\mathbb{k}(x_1, \ldots, x_n)$

G-invariant rational functions: $\mathbb{k}(x_1, \ldots, x_n)^G$

Basic queries:

- What is a small set of generators of $\mathbb{k}(x_1, \ldots, x_n)^G$ as a $\mathbb{k}$-field?
  1st fundamental theorem
- What are the relations between these generators?
  2nd fundamental theorem

e.g. $S_2 = \{\mathrm{id}, \tau\}$ act on $\mathbb{k}(x, y)$ as $\tau(x) = x$, $\tau(y) = \frac{x^3+x}{y}$

Problems of Noether and Lüroth

rational = isomorphic to $\mathbb{k}(y_1, \dots, y_m)$ for some m

Noether's problem[1913] (original): let a finite group G act on $\mathbb{k}^n$ by permuting the coordinates.

Is $\mathbb{k}(x_1, \dots, x_n)^G$ rational?

Problems of Noether and Lüroth

rational = isomorphic to $\mathbb{k}(y_1, \ldots, y_m)$ for some m

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Is $\mathbb{k}(x_1, \ldots, x_n)^G$ rational?

Lüroth's theorem[1876]: every $\mathbb{k}$-subfield of $\mathbb{k}(x)$ is rational.

Lüroth's problem: is every $\mathbb{k}$-subfield K of $\mathbb{k}(x_1, \ldots, x_n)$ rational?

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Noether's problem[1913]: let a finite group G act linearly on $\Bbbk^n$

Is $\Bbbk(x_1, \ldots, x_n)^G$ rational?

NO Swan[69]; Clemens-Griffiths[72], Artin-Mumford[72]

YES if G generated by reflections Chevalley–Shephard–Todd[54]

char $\Bbbk \nmid |G|$

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Lüroth's problem: is every $\Bbbk$-subfield K of $\Bbbk(x_1, \ldots, x_n)$ rational?

YES if $n = 2$ and $\Bbbk = \mathbb{C}$, or $\operatorname{trdeg}_{\Bbbk} K = 1$

NO in general

Free algebra

NC analog of polynomials $\mathbb{k}[x_1, \ldots, x_n]$:

free algebra of noncommutative polynomials of rank n

$\mathbb{k}\!<\!x_1, \ldots, x_n\!>$

e.g.

$$x_1 x_2 x_1^2 - x_2 x_1 + x_1 x_2 - 3$$

Free skew field

60s/70s: Schützenberger, Amitsur, Cohn

NC analog of rational functions $\mathbb{k}(x_1, \ldots, x_n)$:

free skew field of noncommutative rational functions of rank n

$\mathbb{k}\langle\!\!\!\!\!(x_1, \ldots, x_n)\!\!\!\!\!\rangle$

e.g.

$$(x_1 - x_2 x_1^{-1} x_2)^{-1}$$

$$= -x_2^{-1} x_1 (x_2 - x_1 x_2^{-1} x_1)^{-1}$$

$1 - x_1(x_2x_1)^{-1}x_2 = 0$, so $(1 - x_1(x_2x_1)^{-1}x_2)^{-1}$ undefined

Some peculiarities

"dimension" / "transcendence degree"

embeddings don't increase ranks

$$\Bbbk\langle x_0, x_1, x_2, \ldots \rangle \hookrightarrow \Bbbk\langle x, y \rangle, \quad \Bbbk(\!\langle x_0, x_1, x_2, \ldots \rangle\!) \hookrightarrow \Bbbk(\!\langle x, y \rangle\!)$$

$$x_0 \mapsto y,\ x_1 \mapsto [y, x],\ x_2 \mapsto [[y, x], x],\ \ldots$$

$$[y, x] = yx - xy$$

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$$x_0 \mapsto y, \; x_1 \mapsto [y, x], \; x_2 \mapsto [[y, x], x], \; \ldots$$

$$[y, x] = yx - xy$$

"freeness" in free algebra is weaker than freeness in free skew field

$$x, xy, xy^2$$

do not satisfy a nontriv poly relation

but they satisfy a nontriv rat relation $xy^2 - (xy)x^{-1}(xy) = 0$

Some peculiarities

“dimension” / “transcendence degree”

embeddings don’t increase ranks

“freeness” in free algebra is weaker than freeness in free skew field

Good news:

Cohn[85]: $\mathbb{k}<x_1, \ldots, x_m> \cong \mathbb{k}<y_1, \ldots, y_n> \iff m = n$

Schofield[85]: $\mathbb{k}\lt\!\!\!\{ x_1, \ldots, x_m \}\!\!\!\gt \cong \mathbb{k}\lt\!\!\!\{ y_1, \ldots, y_n \}\!\!\!\gt \iff m = n$

Noncommutative polynomial invariants

back to the familiar example

S_2 acts on $\Bbbk\langle x, y\rangle$ by $x \leftrightarrow y$

Wolf[36]: $\Bbbk\langle x, y\rangle^{S_2}$ is a free algebra on the infinite set

$$xy^{n-1} + yx^{n-1} \qquad \text{for } n \in \mathbb{N}$$

Noncommutative polynomial invariants

are almost NEVER finitely generated

Kharchenko[78], Formanek–Dicks[82]:

Let a finite group G act linearly on $\mathbb{k}^n$. Then $\mathbb{k}{<}x_1, \ldots, x_n{>}^G$ is the free algebra on an infinite set

unless the action is scaling by a primitive m^{th} root of unity, in which case it is free on n^m generators

Noncommutative rational invariants

back to the familiar example

S_2 acts on $\Bbbk<x,y>$ by $x \leftrightarrow y$;   assume char $\Bbbk \neq 2$

NC poly invariants: $x+y, (x-y)(x+y)^{n-1}(x-y)$ for $n \in \mathbb{N}$

Agler-Young[14]: these can be rationally expressed with only
$f_{-1} = x+y$, $f_0 = (x-y)^2$, $f_1 = (x-y)(x+y)(x-y)$:

$$(x-y)(x+y)^n(x-y) = f_1\left(f_0^{-1}f_1\right)^{n-1} \quad n = 2, 3, \dots$$

Klep–Podlogar–Pascoe–V[20]: $\Bbbk\langle\!\langle x,y \rangle\!\rangle^{S_2}$ is free on 3 generators

Finite generation of rational invariants

Theorem (Derksen-V[25])

Let a finite group G act on $\Bbbk\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$.

The skew field $\Bbbk\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle^G$ is generated by at most $|G|^2(n-1) + |G|$ elements over $\Bbbk$.

- constructive
- criterion for freeness of generators — we're blind in general!
  $\Bbbk\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle^G$ is "finitely presented"
- don't know of an instance where the bound is attained

Free Noether problem

has a positive solution in the nonmodular case

Theorem (Derksen-V[25])

Let a finite group G act faithfully and linearly on $\Bbbk\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$, and $\operatorname{char} \Bbbk \nmid |G|$.

The skew field $\Bbbk\langle\!\langle x_1, \dots, x_n \rangle\!\rangle^G$ is free of rank $|G|(n-1)+1$.

- Klep-Pascoe-Podlogar-V[20] — for *abelian* and *certain solvable* G
- constructive
- char essential? — not for $G = S_2$
- Nielsen–Schreier index formula for free groups?
  not directly, not a coincidence

Freeness of skew subfields & Cohn's conjecture

free Lüroth problem

Schofield[85] & co: is a skew $\mathbb{k}$-subfield of the free skew field free?

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Conjecture (Cohn[78])

A commutative $\mathbb{k}$-subfield of the free skew field is $\cong \mathbb{k}(t)$.

- commutative subfield is of transcendence degree 1 over $\mathbb{k}$
- related results:
  comm subalg of the free alg is $\cong \mathbb{k}[t]$
  comm complete subalg of the free power series is $\cong \mathbb{k}[\![t]\!]$
  ...

A nonlinear example

negative answer to Schofield

Let char $\Bbbk \neq 2$, and let $f \in \Bbbk[x]$ be a monic cubic polynomial with simple roots

Let $S_2 = \{\mathrm{id}, \tau\}$ act on $\Bbbk(\!\langle x, y \rangle\!)$ as

$$\tau(x) = y^{-1}xy, \quad \tau(y) = y^{-1}f(x)$$

The skew field $\Bbbk(\!\langle x, y \rangle\!)^{S_2}$

- is NOT free,
- is generated by 4 (but not 3) elements

Elliptic curves in the free skew field

Theorem (Derksen-V[25])

Let $f \in \mathbb{k}[x]$ be a monic cubic polynomial with simple roots and the quad coef α. Let $r, s \in \mathbb{k}\langle\!\langle x, y \rangle\!\rangle$ as

$$r = b^2 - c - \alpha \quad \text{and} \quad s = y - xb + br,$$

$$\text{where} \qquad b = (xy - yx)^{-1}(f(x) - y^2),$$
$$c = (xy - yx)^{-1}(x^2y - yx^2).$$

Then, $r, s \notin \mathbb{k}$, $rs = sr$ and $s^2 = f(r)$.

if char $\mathbb{k} \neq 2$, this is an genus-one curve

Invariants of infinite groups

may need ∞ generators

- $\Bbbk$ infinite, $G = \Bbbk^*$ acts on $\Bbbk\langle\langle x, y\rangle\rangle$ by scaling the 1st variable:

$$\Bbbk\langle\langle x, y\rangle\rangle^G = \Bbbk\langle\langle x^{-r} y x^r : r \in \mathbb{Z}\rangle\rangle$$

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- Intriguing example: $\mathrm{GL}_n(\Bbbk)$ acting linearly on $\Bbbk\langle x_1, \ldots, x_n\rangle$

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Podlogar[23]:

$$r = (xy - yx)^{-2}(x^2y^2 - xy^2x - yx^2y + y^2x^2)$$

is GL_2-invariant,

$$r(x, y) = r(\alpha_{11}x + \alpha_{12}y, \alpha_{21}x + \alpha_{22}y) \qquad \forall \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{k})$$

Open questions

- infinite linear reductive/Lie groups

  modular reps of finite groups

- symmetric group: combinatorial free gens?

- does every curve embed into the free skew field?

- is a skew subfield of the free skew field a coproduct of curves?