Holomorphic rigidity: continued

Peter Huxford, Rice University

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Will mostly focus on the case $\mathrm{UConf}_n\mathbb{C} \to \mathrm{UConf}_m\mathbb{C}$.

Mapping class groups

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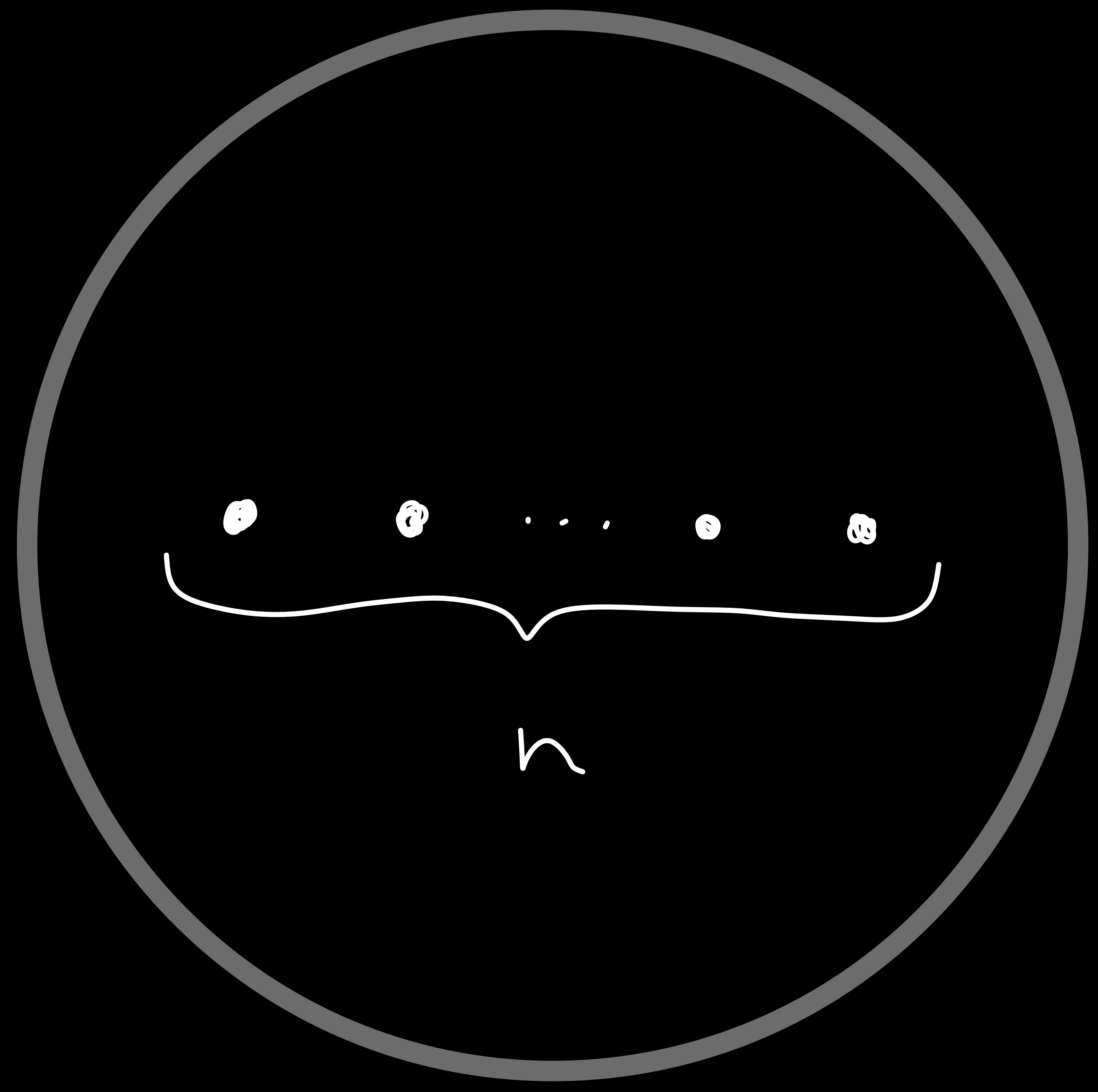
Special case: $\mathrm{Mod}_{1,1} \cong \mathrm{SL}_2\mathbb{Z}$.

Braid group as mapping class group

The braid group is also a mapping class group.



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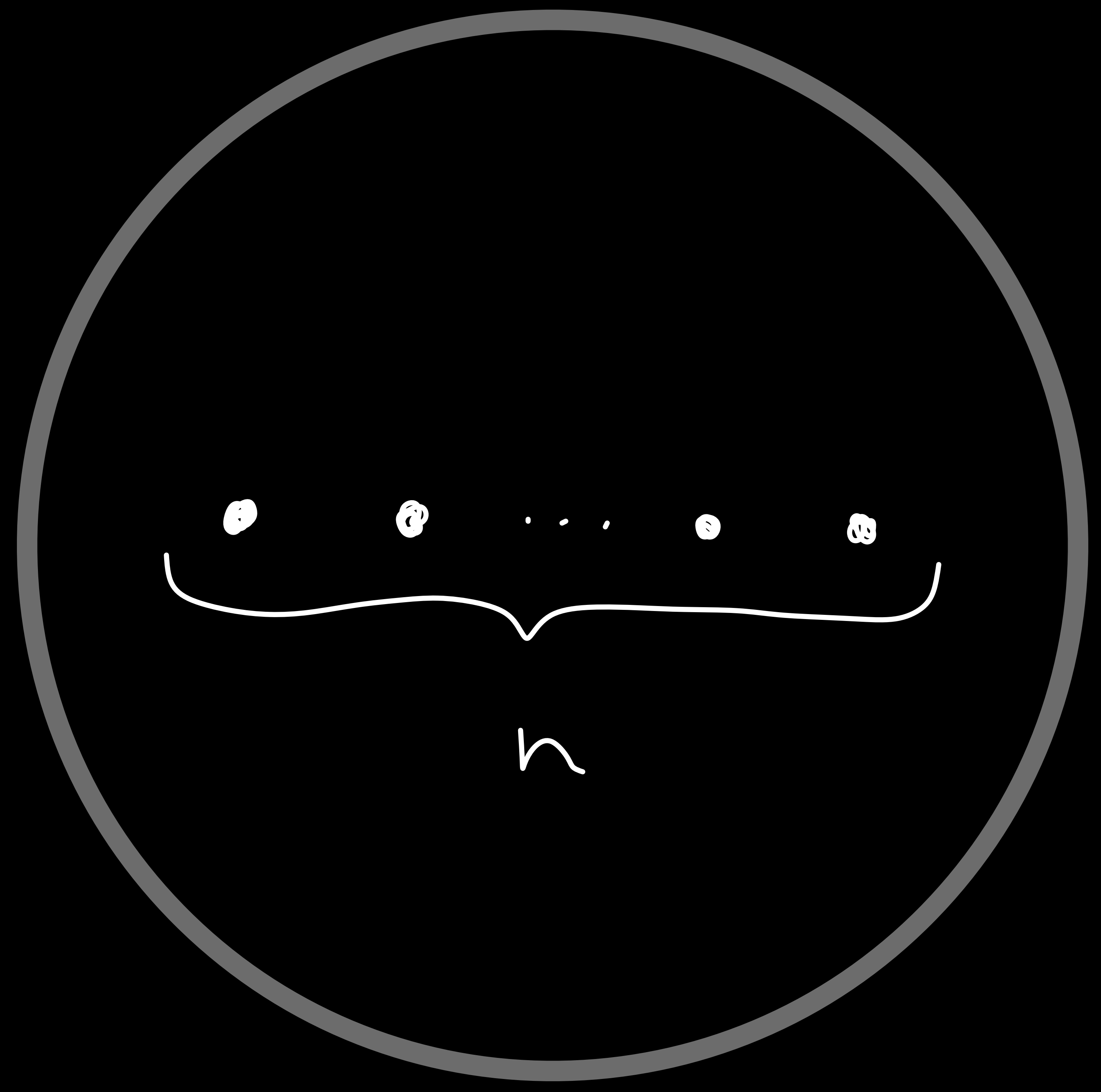
The natural map $B_n \to \mathrm{Mod}_{0,n}^1$ is an isomorphism.



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Universal cover of moduli space

There is an isomorphism of complex-analytic orbifolds

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We can lift any holomorphic map $\mathscr{M}_{g,n} \to \mathscr{M}_{h,m}$ to a holomorphic map $\mathscr{T}_{g,n} \to \mathscr{T}_{h,m}$ between the universal covers.

Monodromy of families

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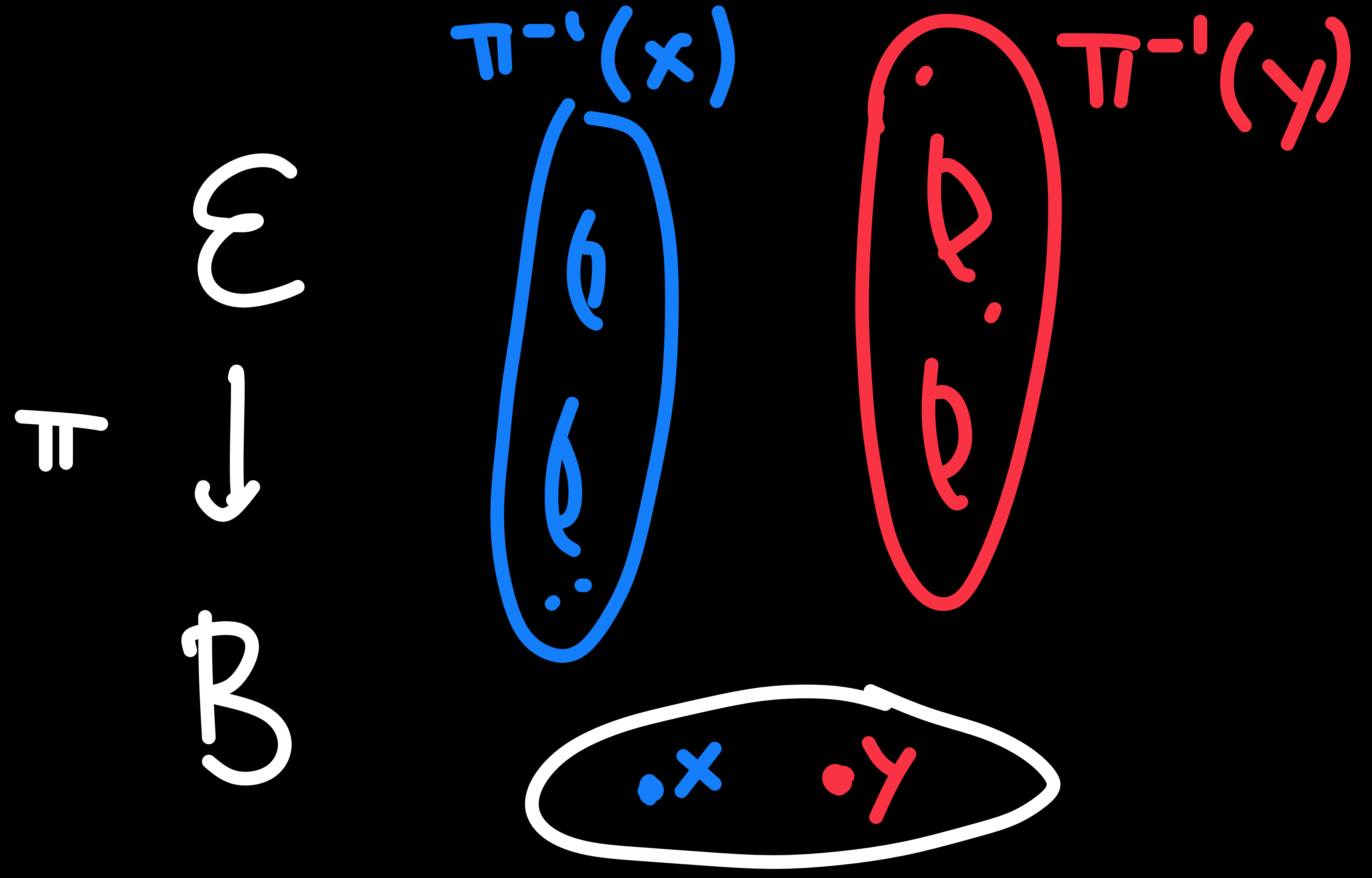

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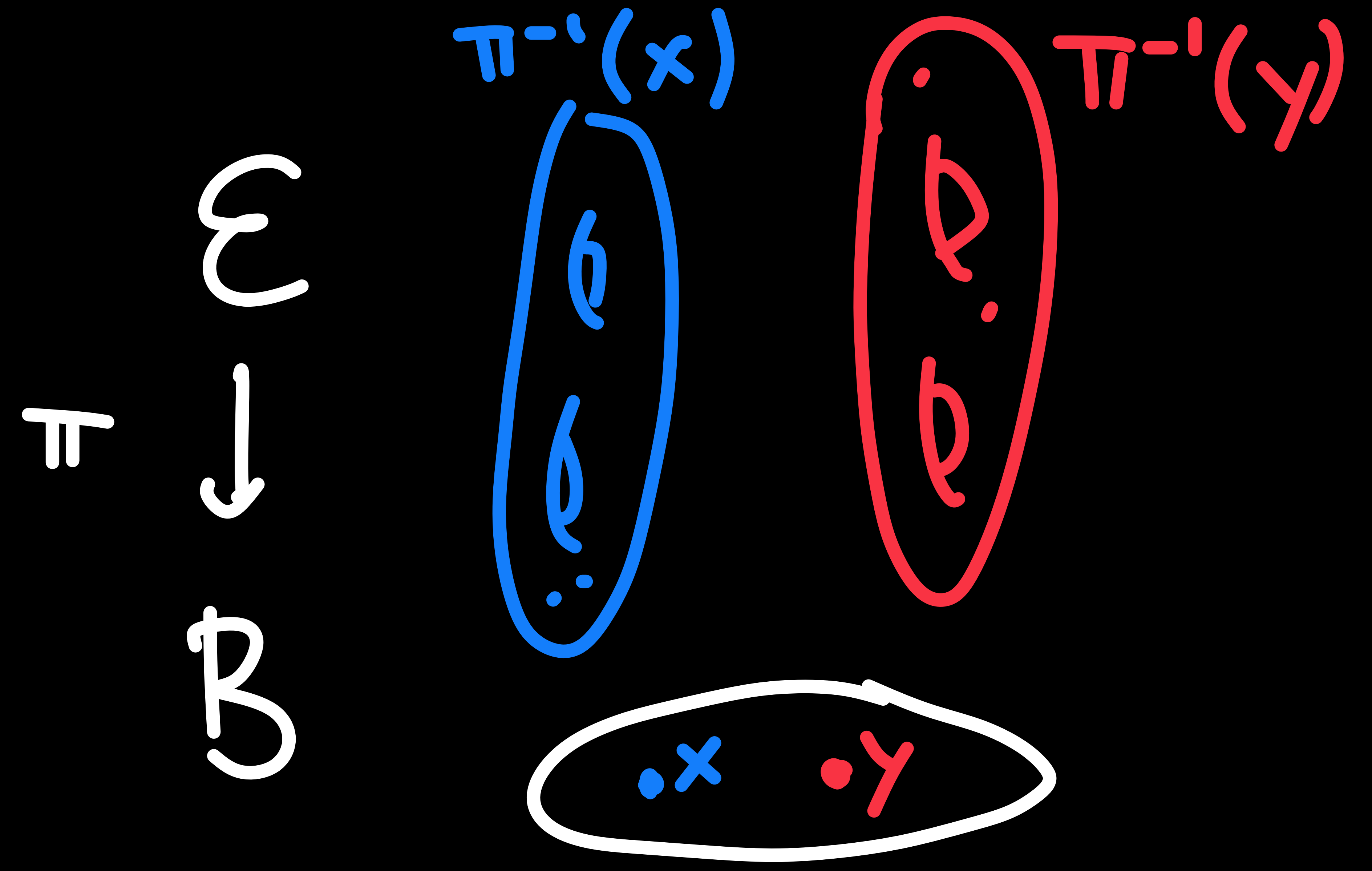

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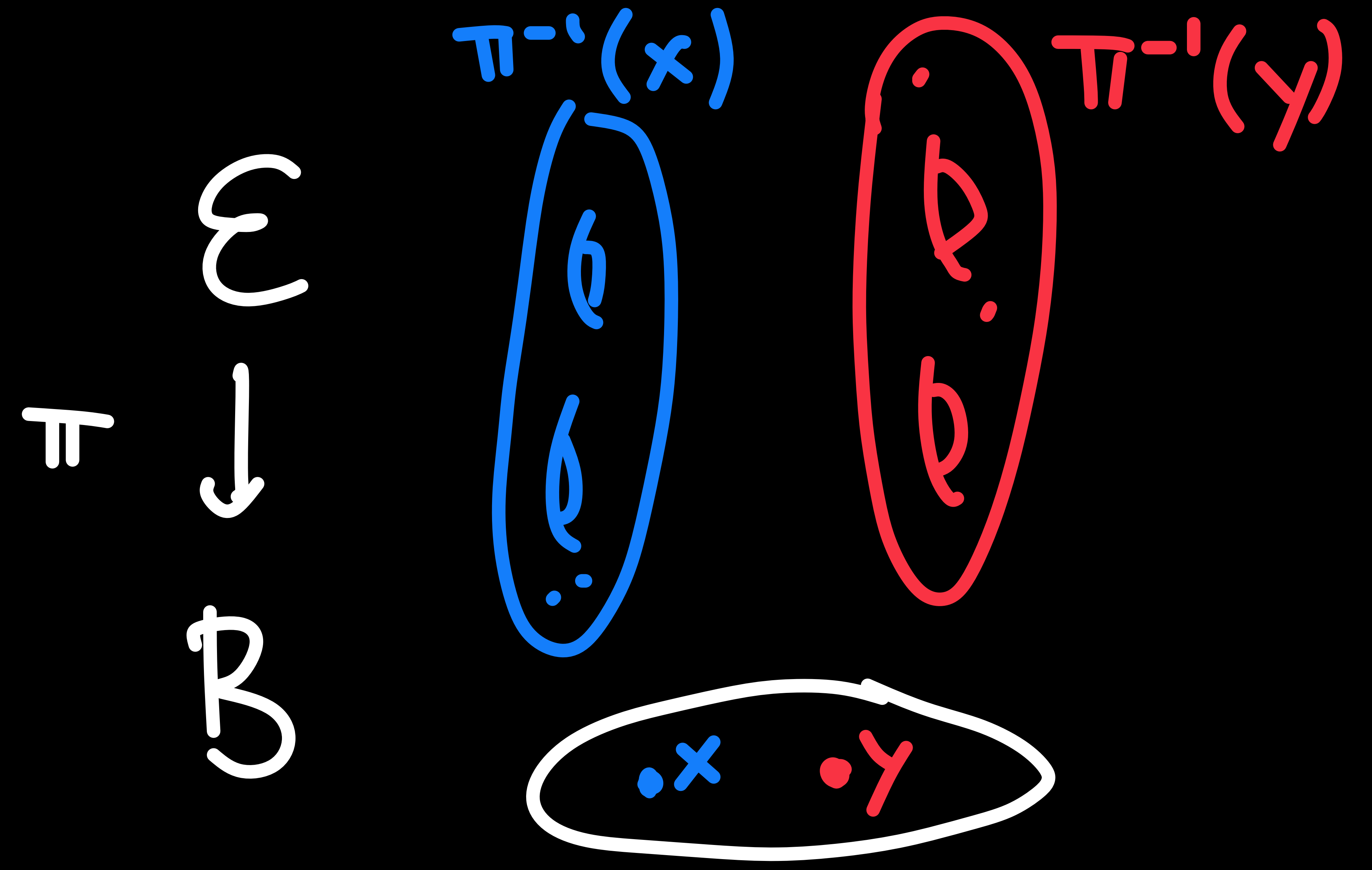

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This produces a well defined homomorphism $\pi_1(B) \to \mathrm{Mod}_{g,n}$, called *monodromy*.

It is the same thing as the induced map on π_1 of the original map.

Finiteness theorems

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Theorem (Imayoshi—Shiga ‘88): True.

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More generally **(Antonakoudis—Aramayona—Souto '18)** show a similar result for B any quasi-projective variety.

Corollary: Any two holomorphic maps $\mathrm{UConf}_n\mathbb{C} \to \mathrm{UConf}_m\mathbb{C}$ that induce the same homomorphism $B_n \to B_m$ have induced maps $\mathrm{UConf}_n\mathbb{C} \to (\mathrm{UConf}_m\mathbb{C})/\mathrm{Aff}$ that are either constant, or exactly equal.

Summary

Two holomorphic maps $f, g\colon \mathrm{UConf}_n\mathbb{C} \to \mathrm{UConf}_m\mathbb{C}$ are *affine equivalent* if they define the same map $\mathrm{UConf}_n\mathbb{C} \to (\mathrm{UConf}_m\mathbb{C})/\mathrm{Aff}$.

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However in general there are non-trivial homomorphisms $\pi_1(\mathscr{M}) \to \pi_1(\mathscr{N})$ which are not induced by holomorphic maps.

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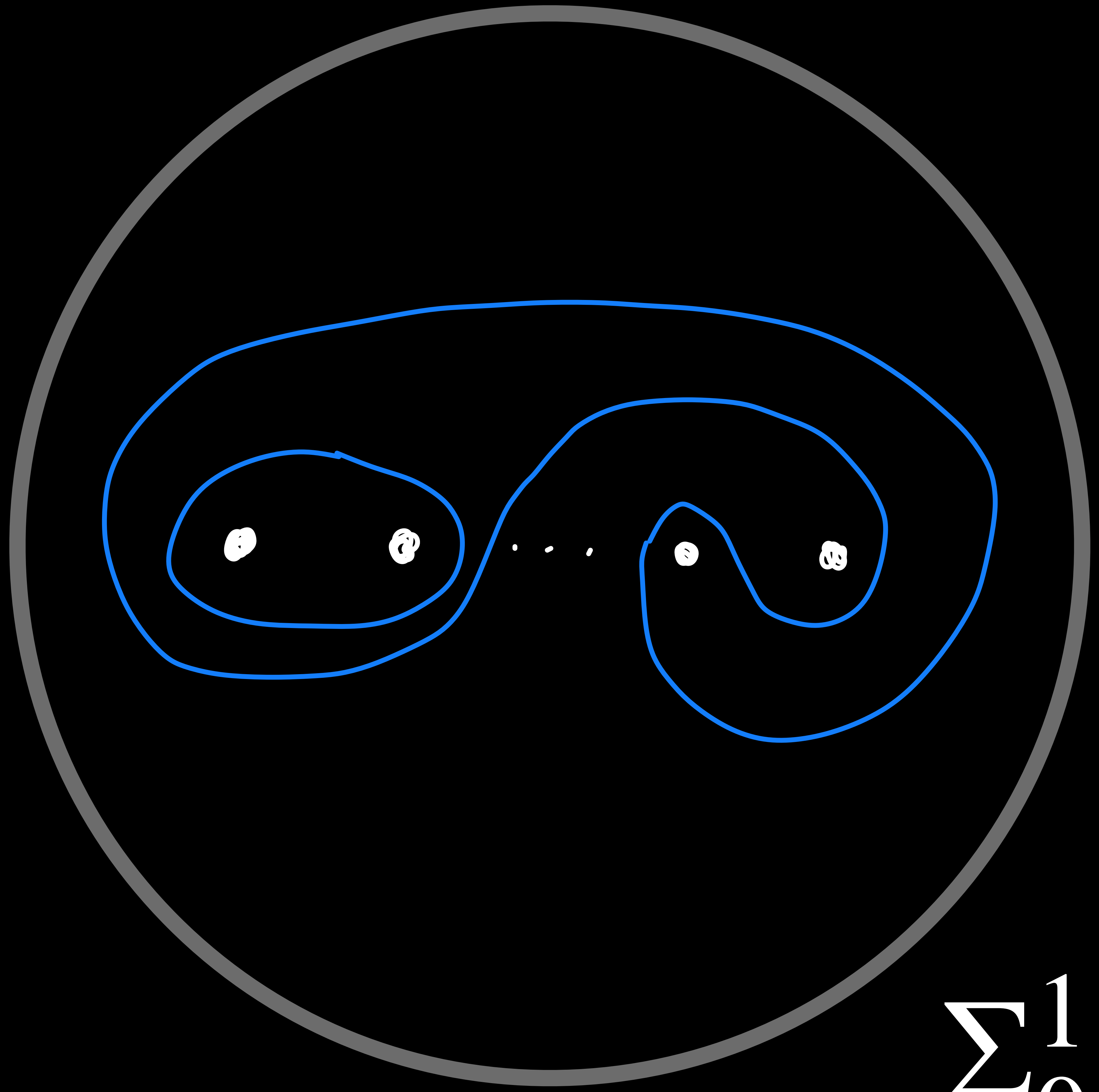

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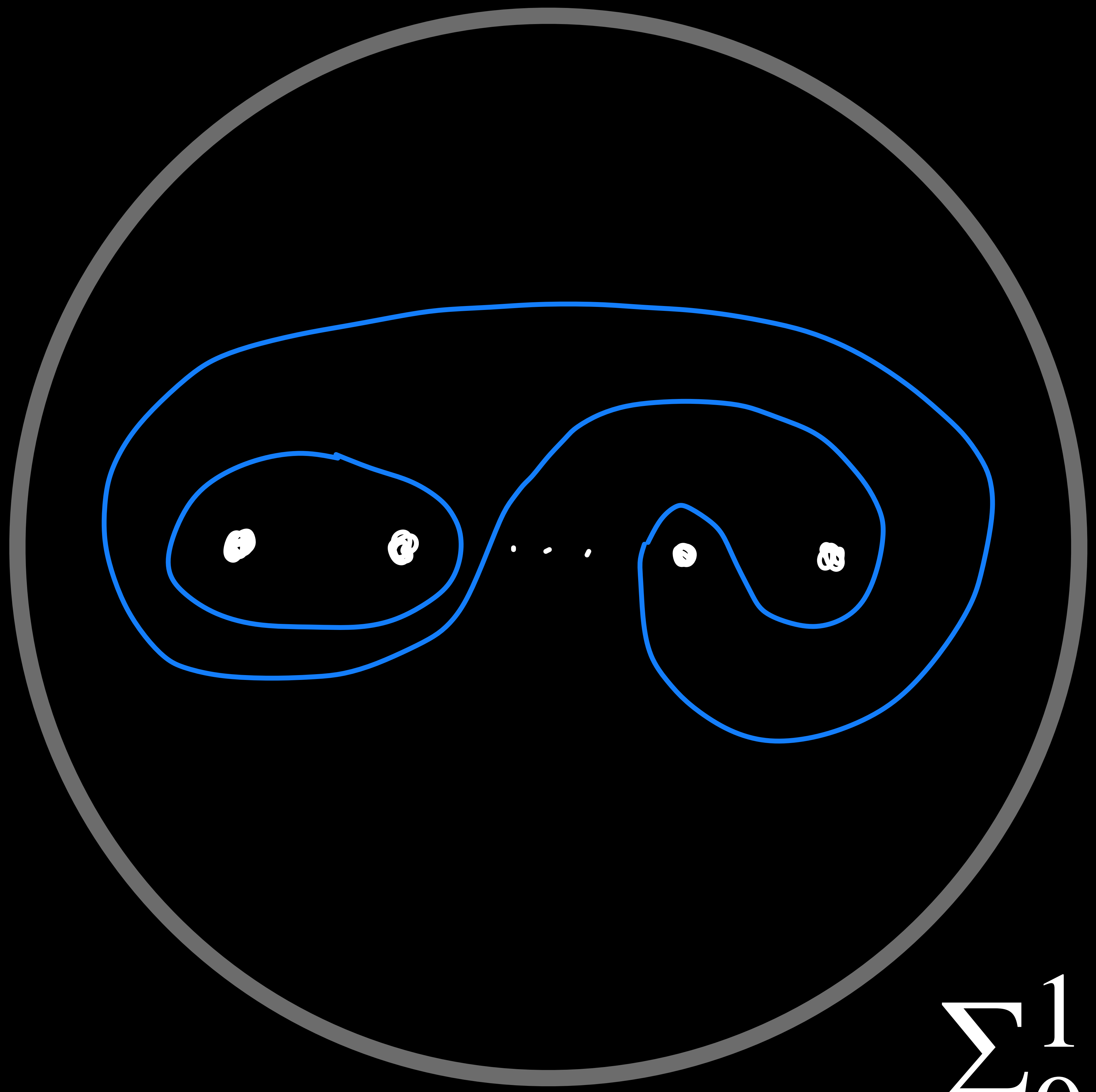

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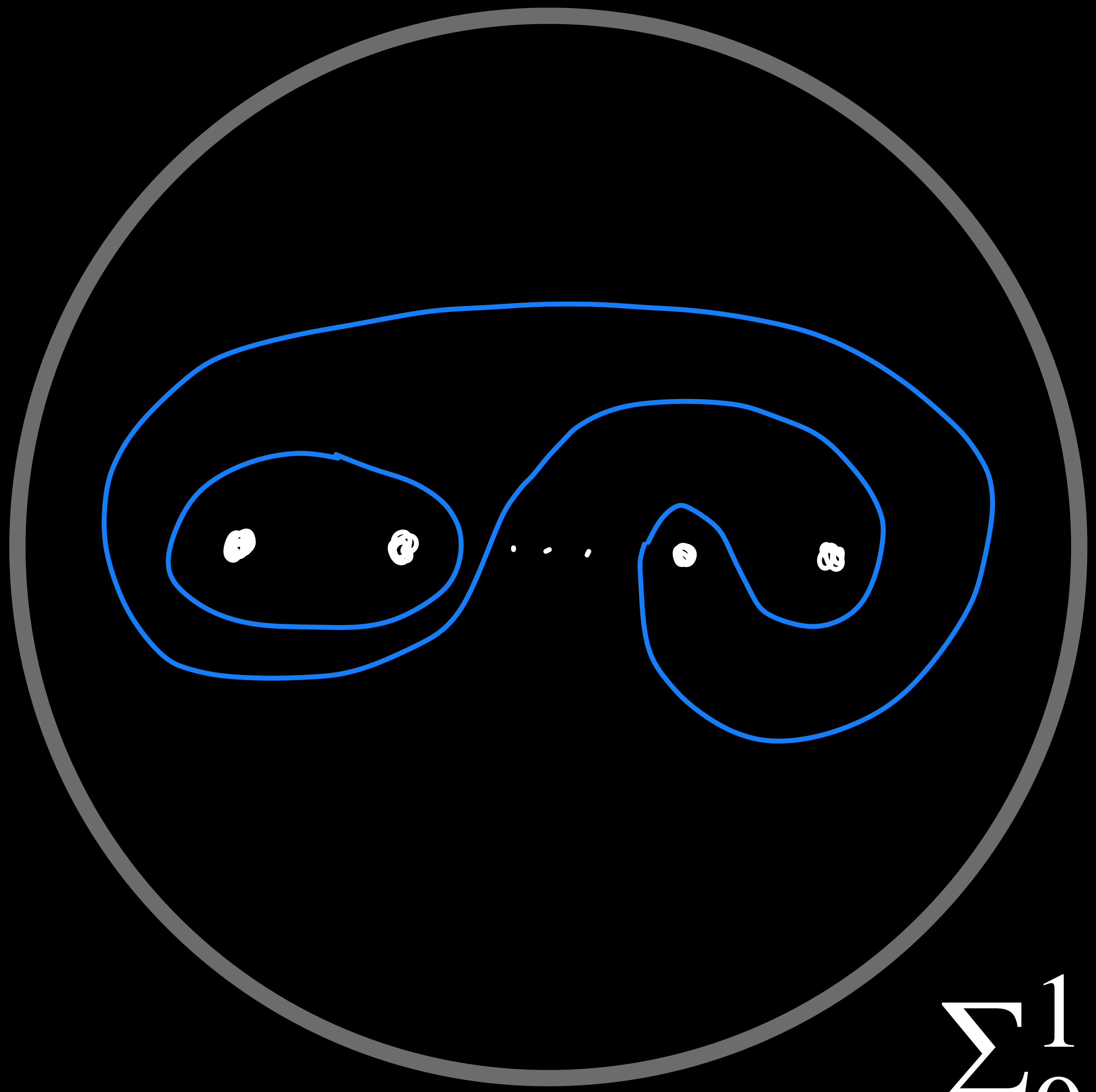

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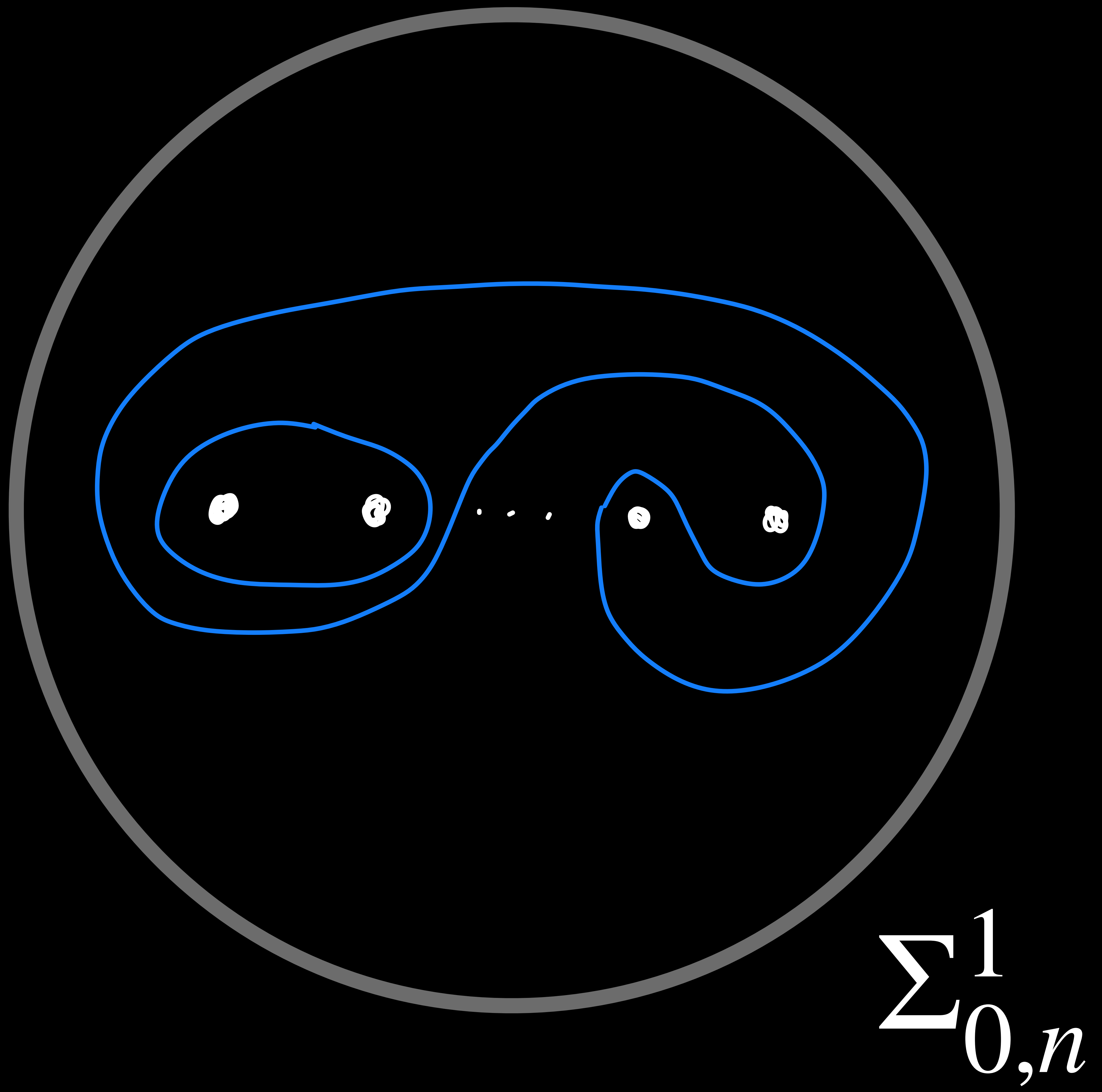

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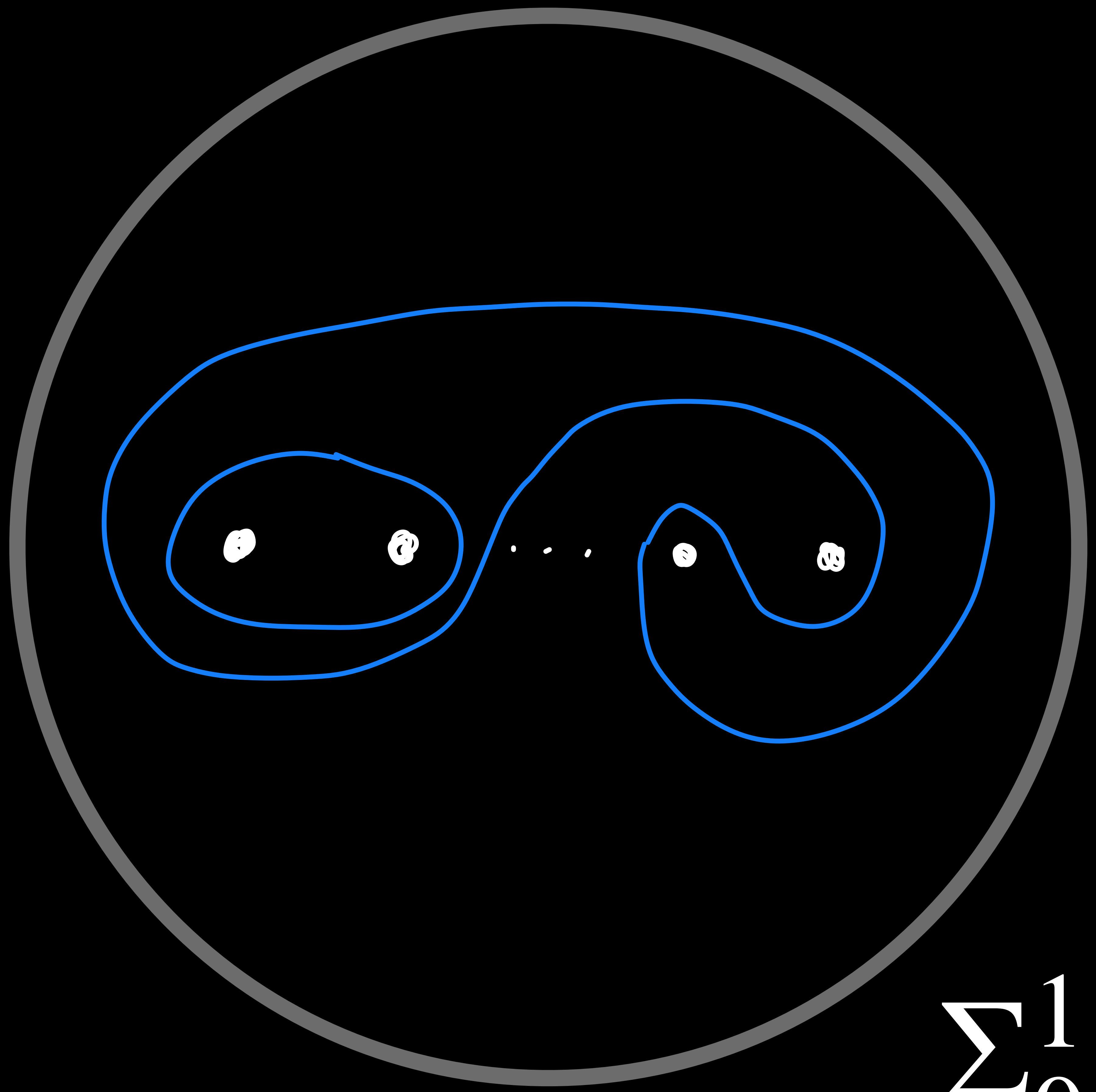

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Theorem (de Pool—Souto, ‘24): If M is an irreducible quasi-projective variety, and $F\colon M \to \mathscr{M}_{g,n}$ is a non-constant holomorphic map, then $F_*\colon \pi_1(M) \to \mathrm{Mod}_{g,n}$ is irreducible.

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Corollary: A holomorphic map $f\colon \mathrm{UConf}_n\mathbb{C} \to \mathrm{UConf}_m\mathbb{C}$ must induce an irreducible homomorphism $f_*\colon B_n \to B_m$.

Other results

Theorem (Chen—Kordek—Margalit, ‘23): If $n \geq 5$, and $m \leq 2n$, and $\varphi\colon B_n \to B_m$ is irreducible and non-cyclic, then $m = n$ and φ is center equivalent to the identity.

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They conjecture the $m \leq 2n$ bound is not necessary.

Theorem (Chen—Salter, '23): If $n \geq 5$, and $m \leq 2n$, then every holomorphic map $\mathrm{UConf}_n\mathbb{C} \to \mathrm{UConf}_m\mathbb{C}$ is affine equivalent to the identity or a constant map.

What about n = 3 or 4?

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where $\alpha = \sigma_1\sigma_2$ and $\beta = \sigma_1\sigma_2\sigma_1$.

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We can precompose each of these with Ferrari's map $R_*\colon B_4 \to B_3$ to obtain a wealth of irreducible homomorphisms $B_4 \to B_3$.

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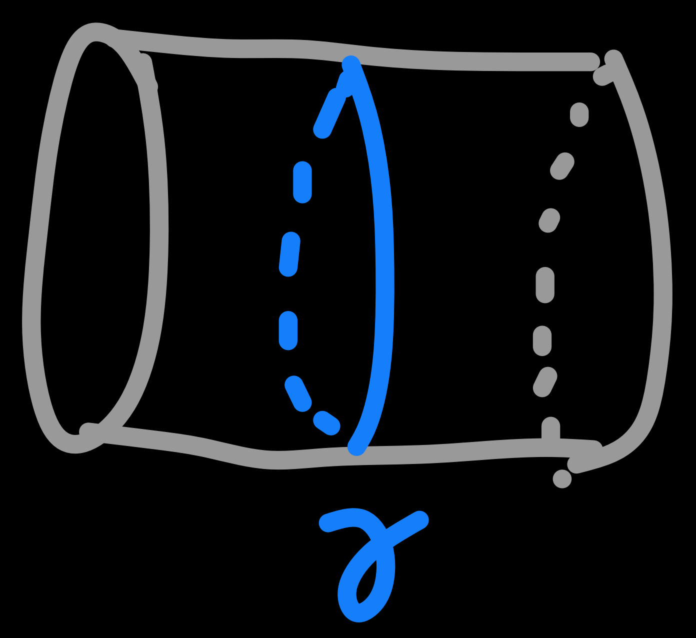

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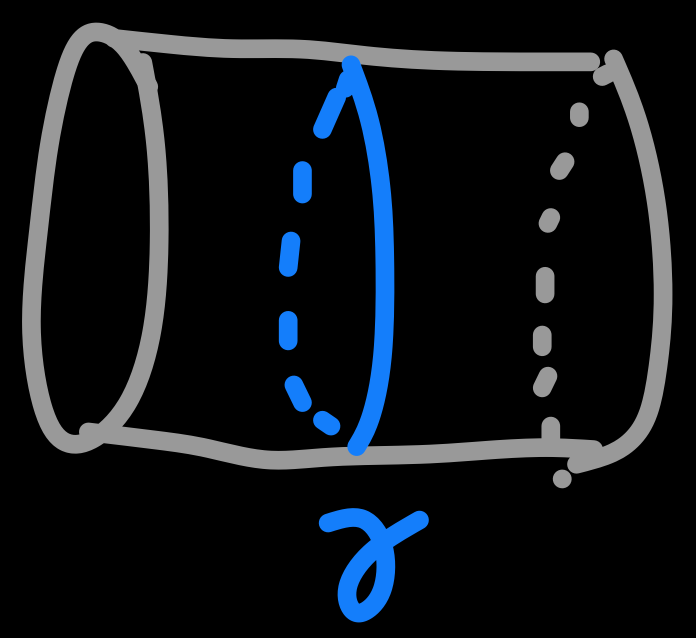

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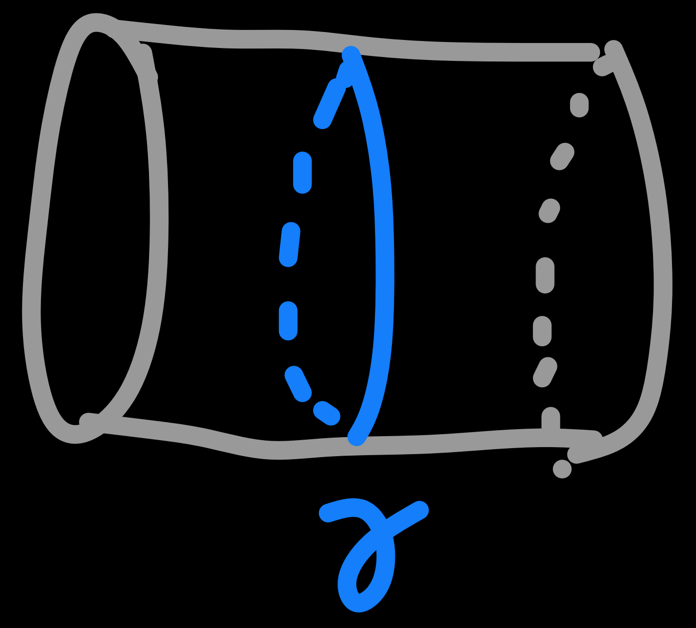

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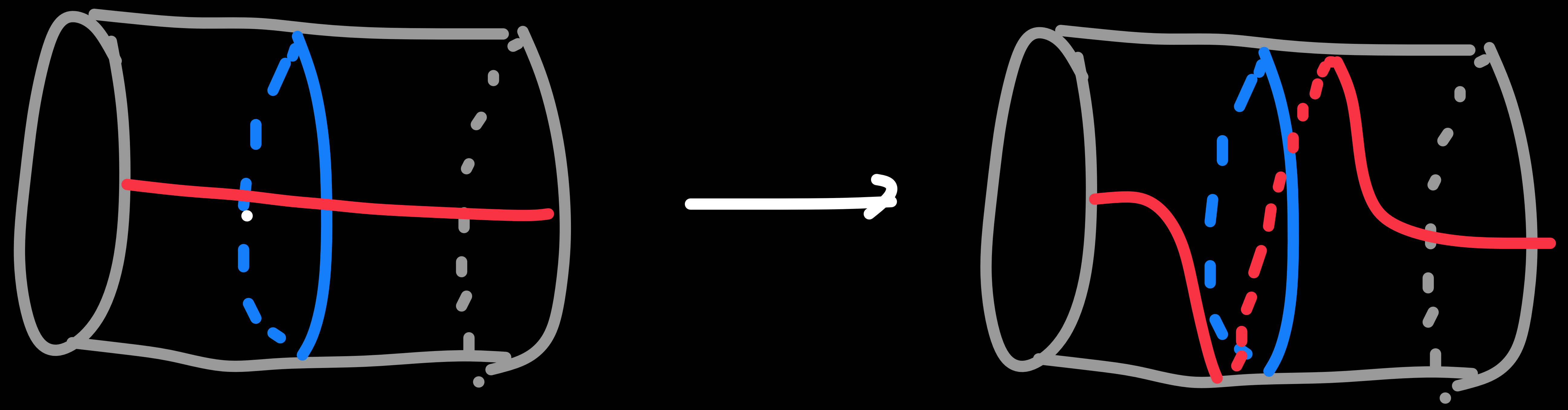
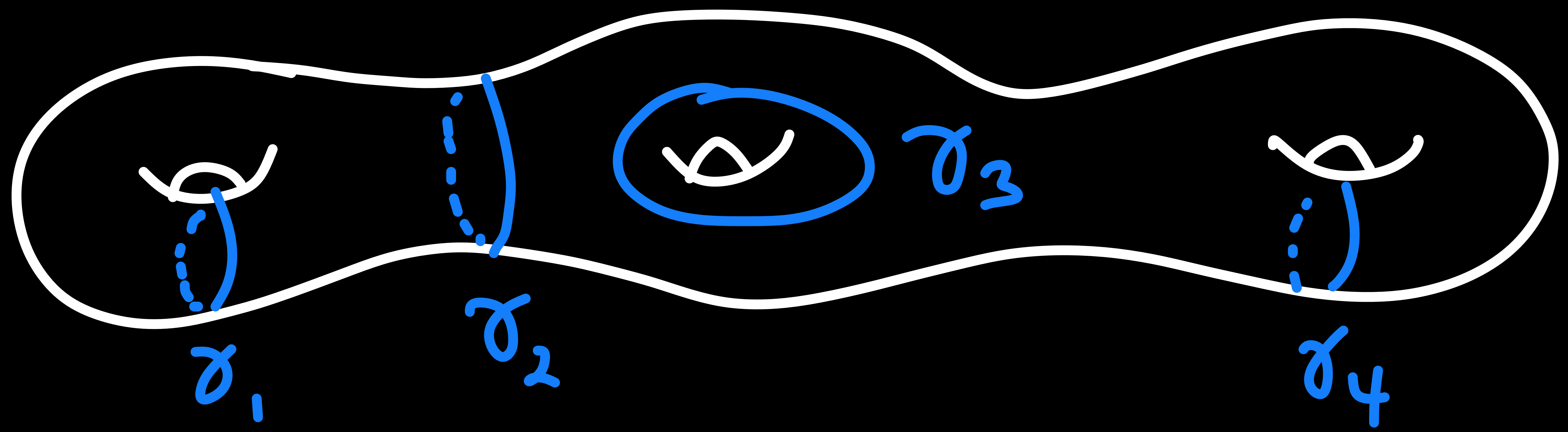


Multitwists

If we let $\gamma_1, \ldots, \gamma_n$ be the curves in a multicurve on an orientable surface Σ, then the Dehn twists $T_{\gamma_1}, \ldots, T_{\gamma_n}$ all commute in $\mathrm{Mod}(\Sigma)$.



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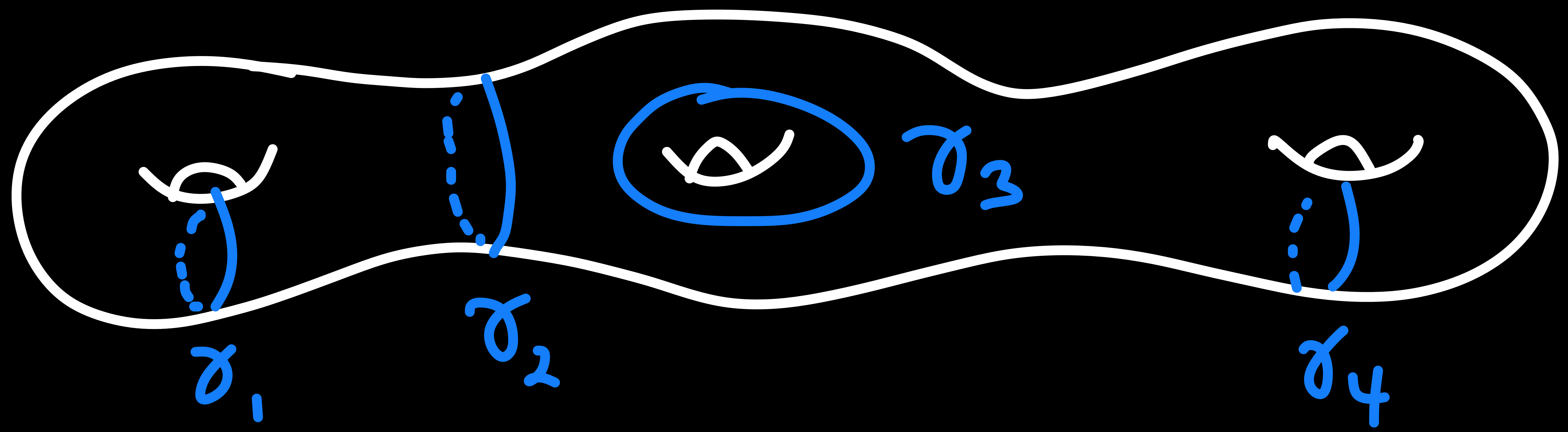
We call an element of the subgroup $\langle T_{\gamma_1}, \ldots, T_{\gamma_n} \rangle \cong \mathbb{Z}^n$ a *multitwist*.



Teichmüller metric

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Theorem (Bers, '78) $\tau(f) = 0$ if and only if f^k is a multitwist for some $k \geq 1$.

Image of standard braid generators

Proposition: If $f\colon \mathrm{UConf}_n\mathbb{C} \to \mathrm{UConf}_m\mathbb{C}$ is holomorphic, and $f_*\colon B_n \to B_m$ is the induced homomorphism, then $f_*(\sigma_i)$ is a root of a multitwist in B_m, for all $1 \leq i < n$.

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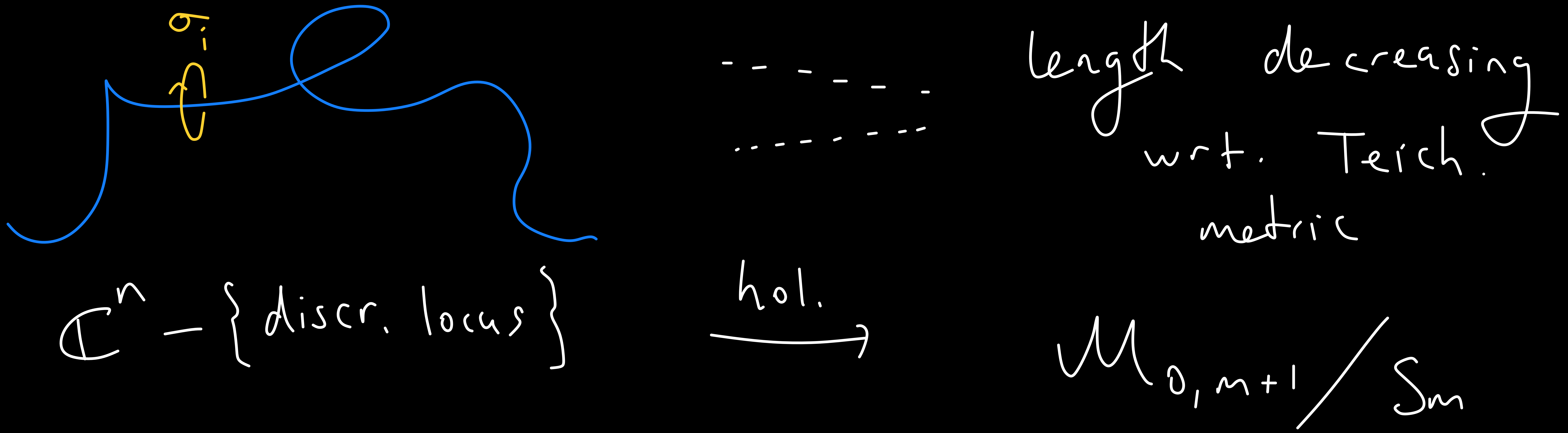
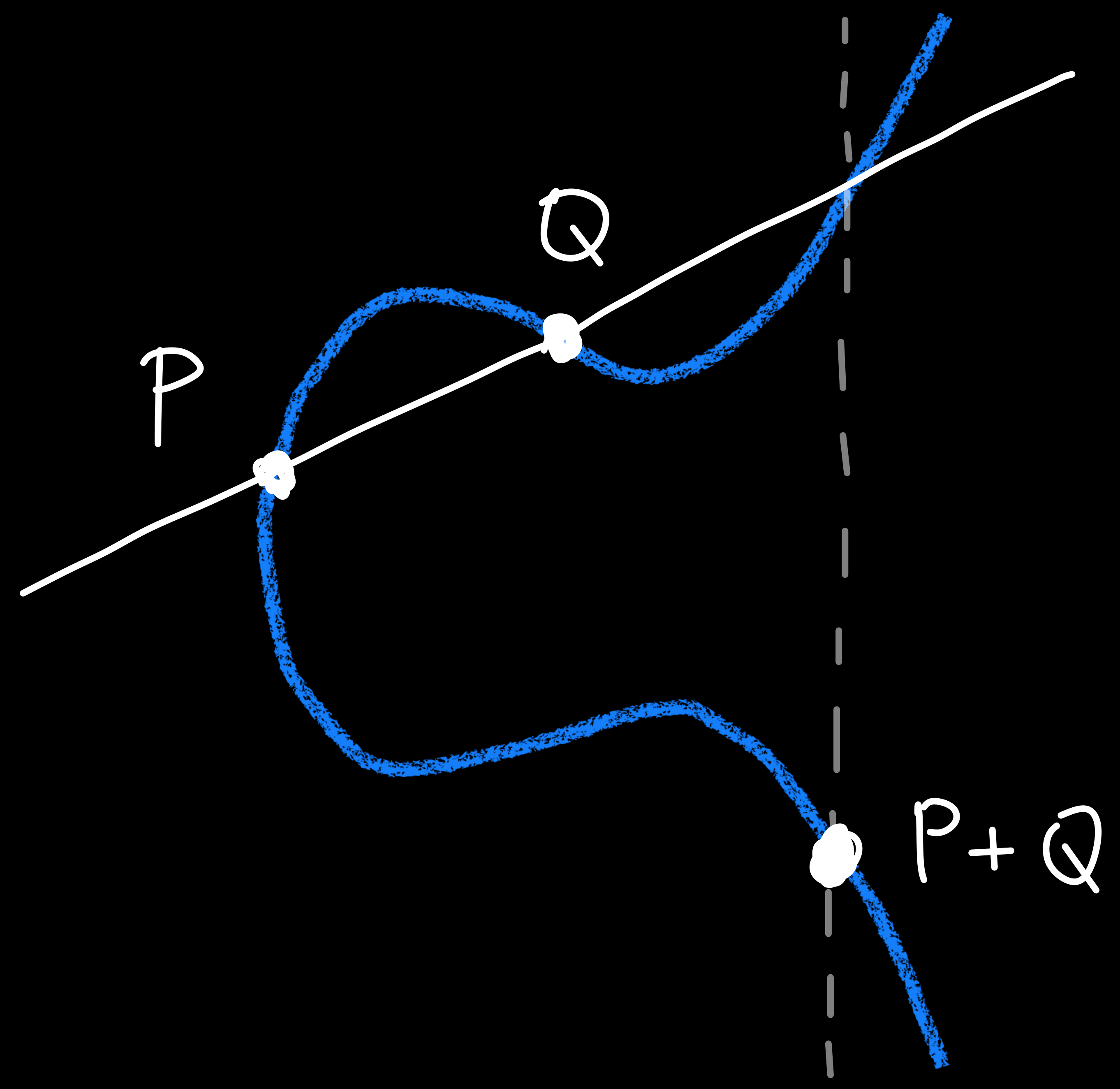
Proof sketch:



Elliptic curve constructions

Given $\{x_1, x_2, x_3\} \in \mathrm{UConf}_3\mathbb{C}$, can construct an *elliptic curve*

$$y^2 = (x - x_1)(x - x_2)(x - x_3).$$



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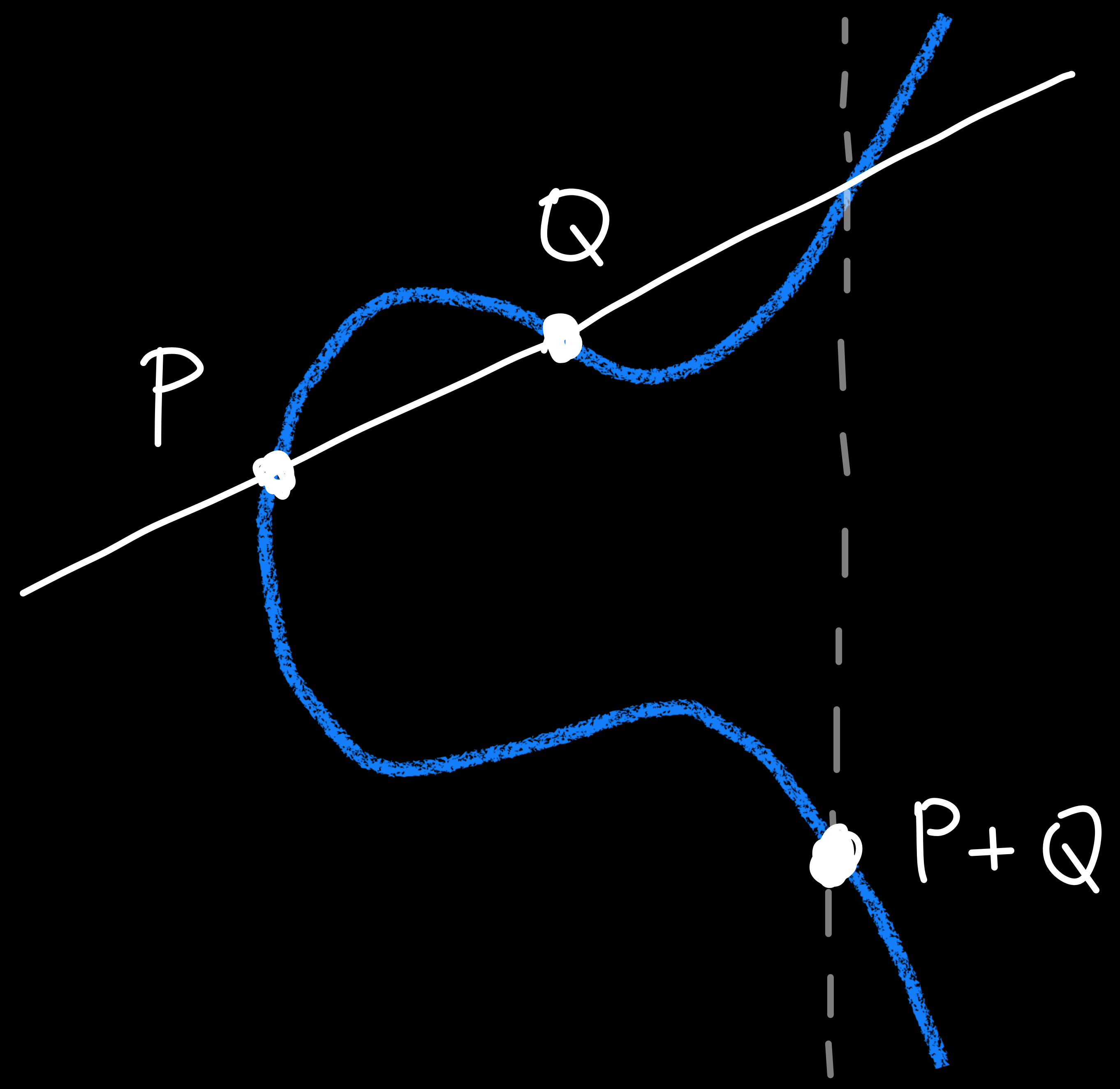

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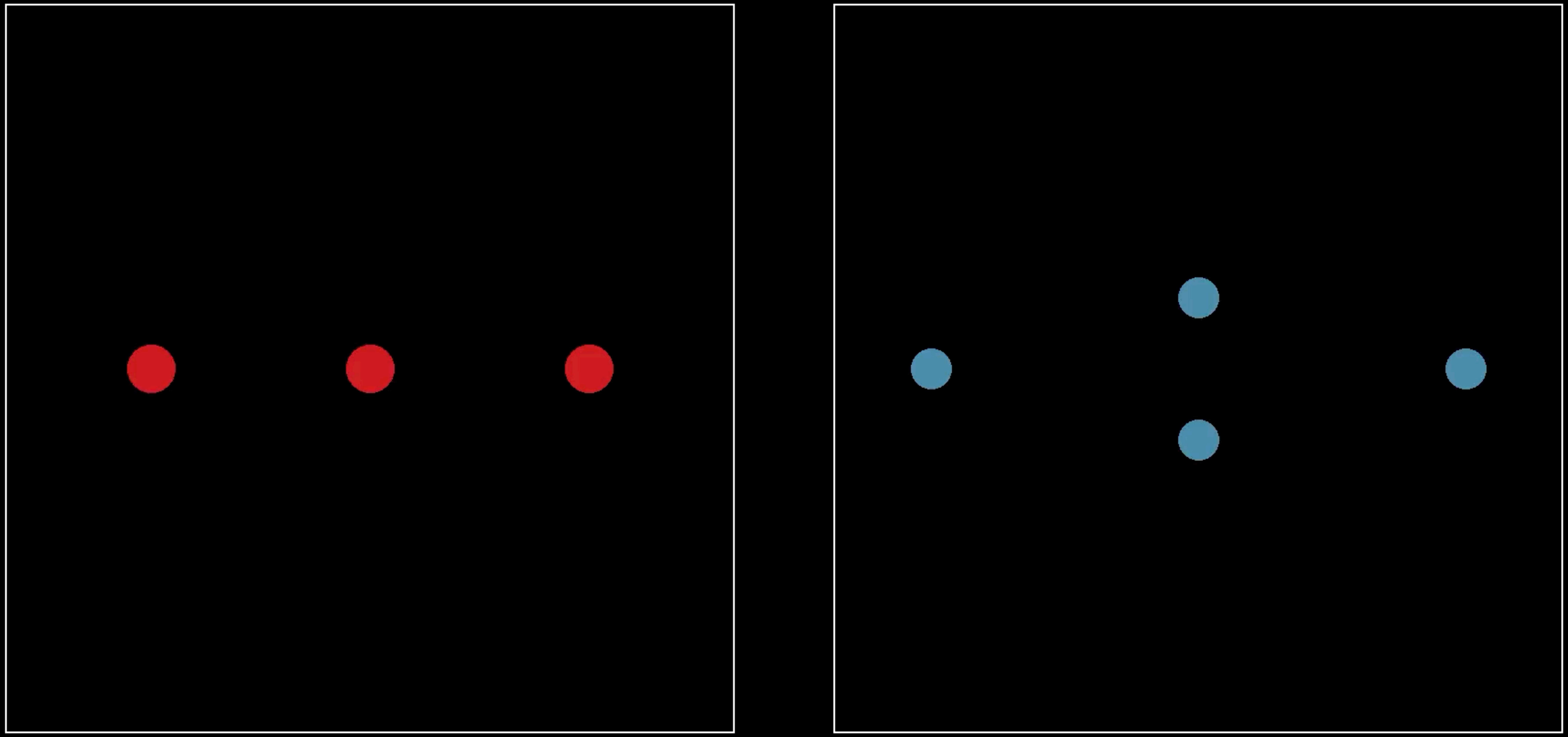
Get a holomorphic map $\Psi_k\colon \mathrm{UConf}_3\mathbb{C} \to \mathrm{UConf}_{m_k}\mathbb{C}$ by

$$\Psi_k(\{x_1, x_2, x_3\}) = \{x\text{-coordinates of points of order } k\}$$

Example: $\Psi_3 \colon \mathrm{UConf}_3\mathbb{C} \to \mathrm{UConf}_4\mathbb{C}$



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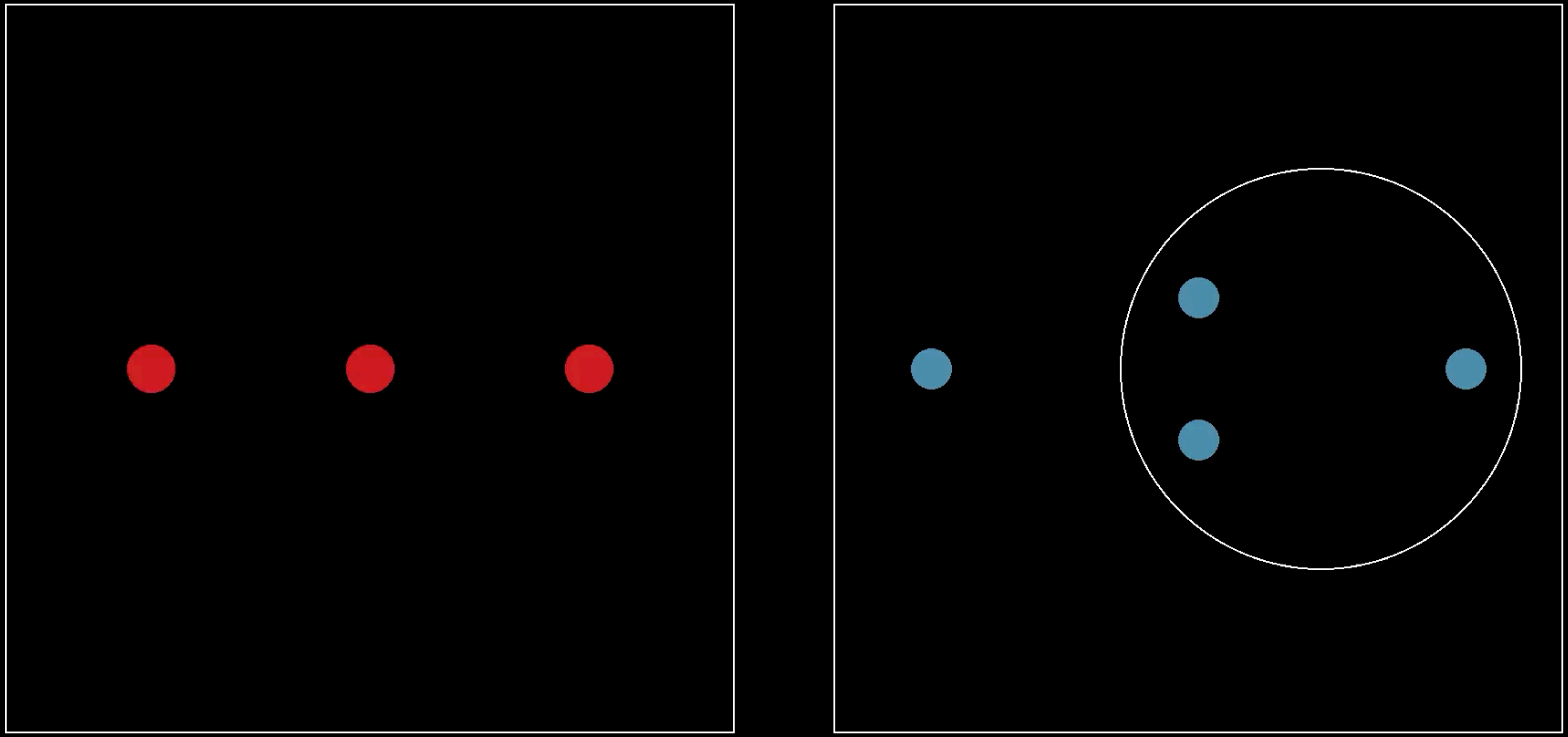
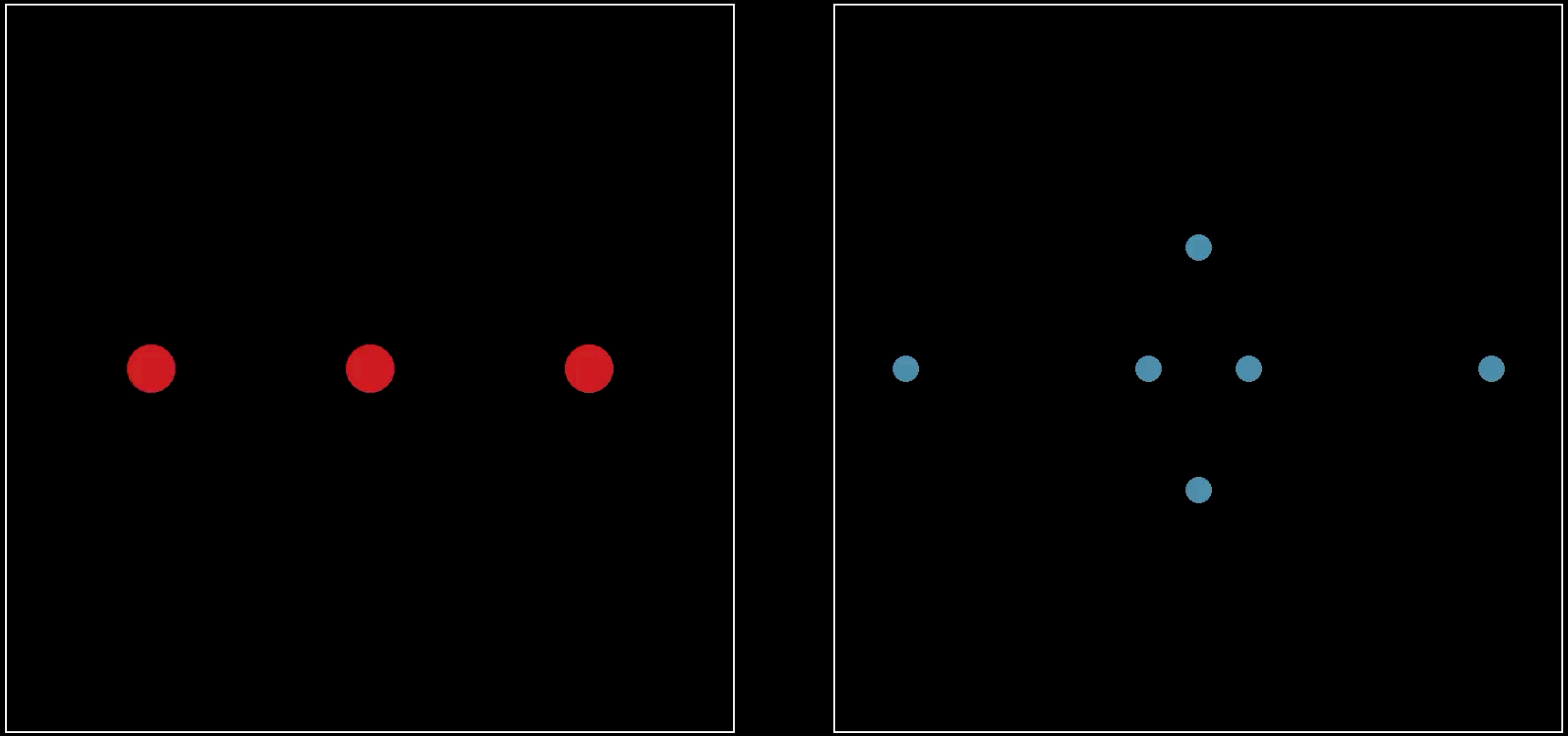


Theorem: (H—Schillewaert, '23)

If f: $\mathrm{UConf}_n\mathbb{C} \to \mathrm{UConf}_m\mathbb{C}$ is holomorphic, $n \geq 3$, and $m \leq \max\{n,4\}$, then it is equivalent to one of the following

- a constant map,
- the identity map,
- $R\colon \mathrm{UConf}_4\mathbb{C} \to \mathrm{UConf}_3\mathbb{C}$
- $\Psi_3\colon \mathrm{UConf}_3\mathbb{C} \to \mathrm{UConf}_4\mathbb{C}$
- $\Psi_3 \circ R\colon \mathrm{UConf}_4\mathbb{C} \to \mathrm{UConf}_4\mathbb{C}$

Example: $\Psi_4\colon \mathrm{UConf}_3\mathbb{C} \to \mathrm{UConf}_6\mathbb{C}$



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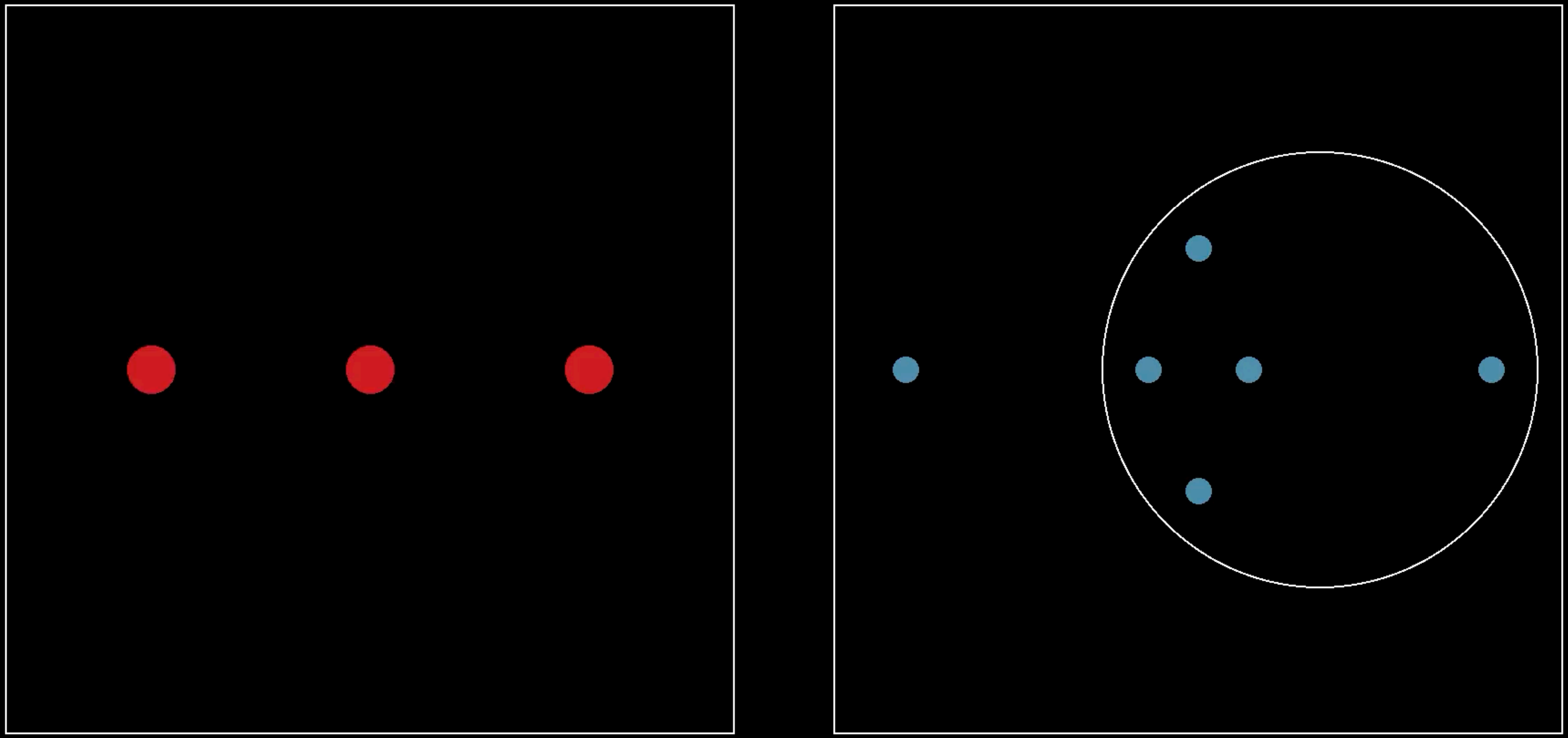
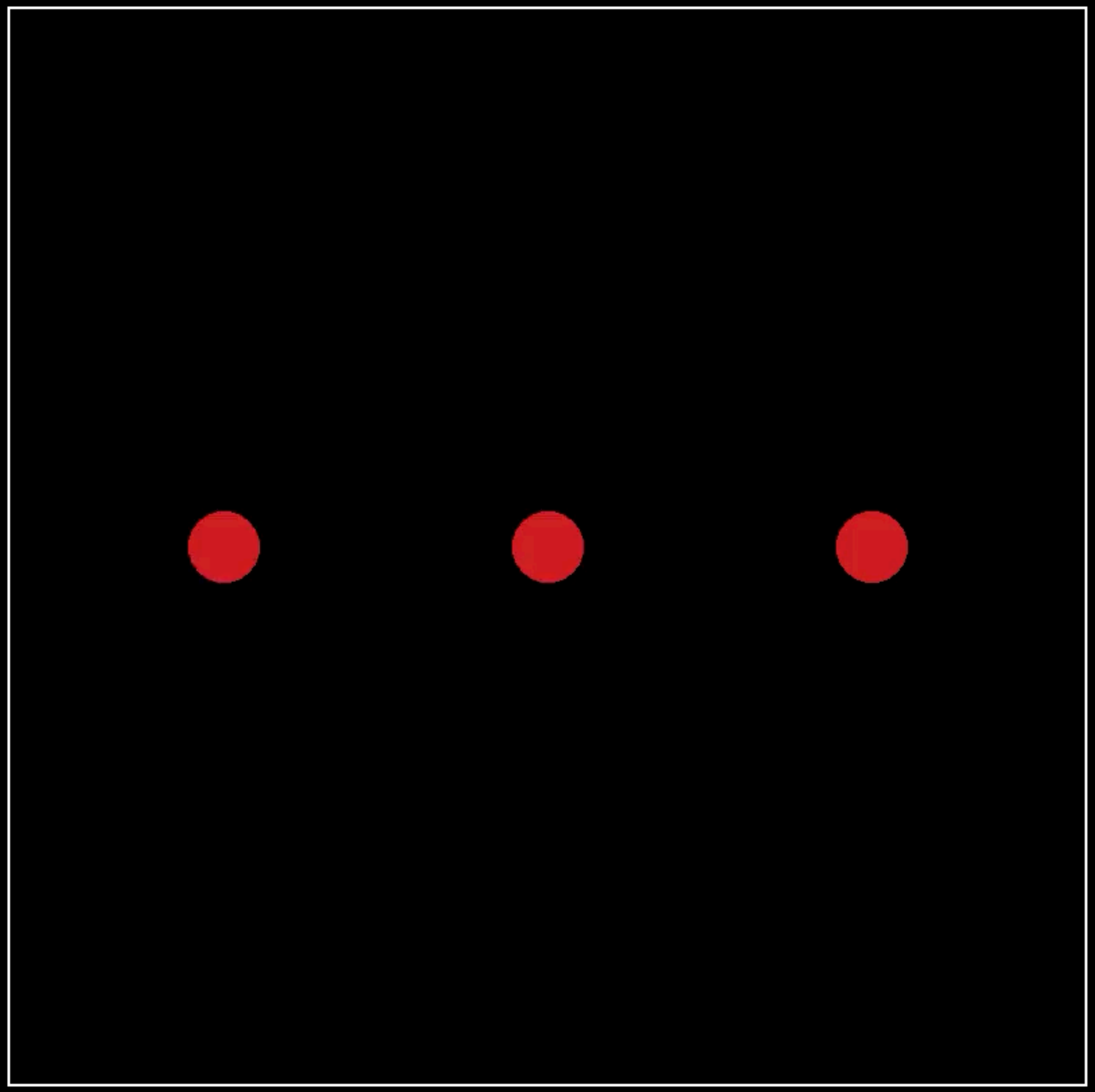
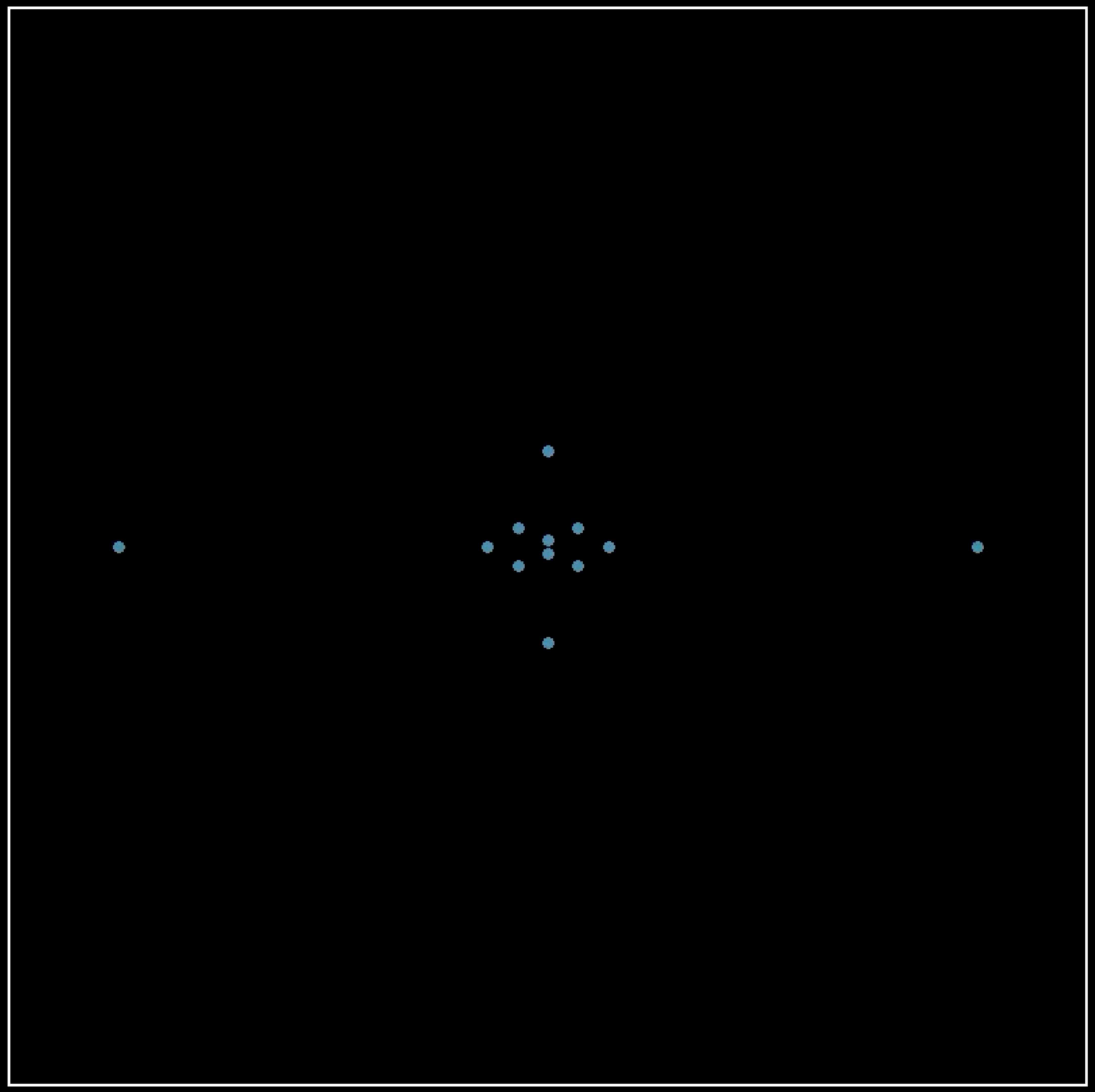


Example: $\Psi_5 \colon \mathrm{UConf}_3\mathbb{C} \to \mathrm{UConf}_{12}\mathbb{C}$





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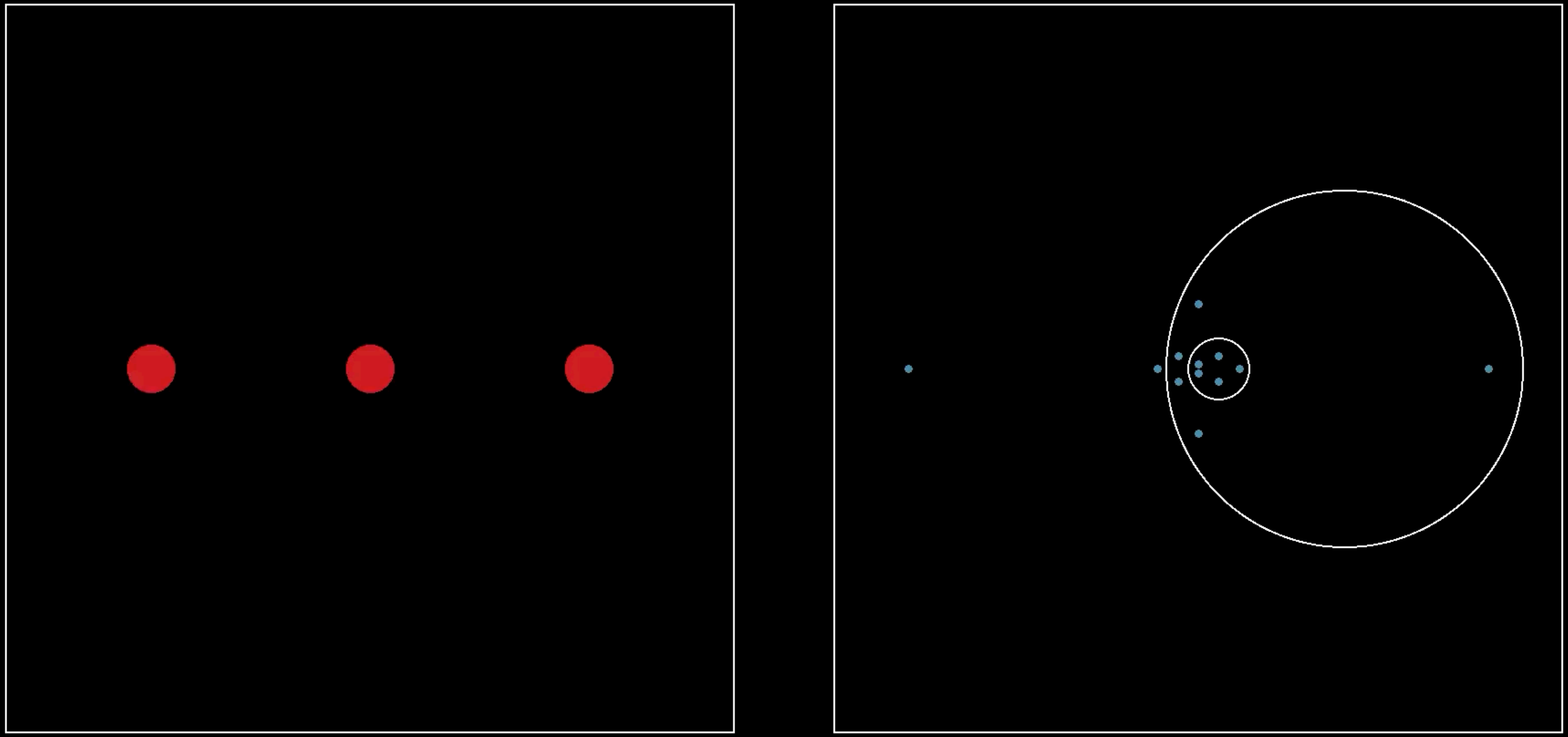

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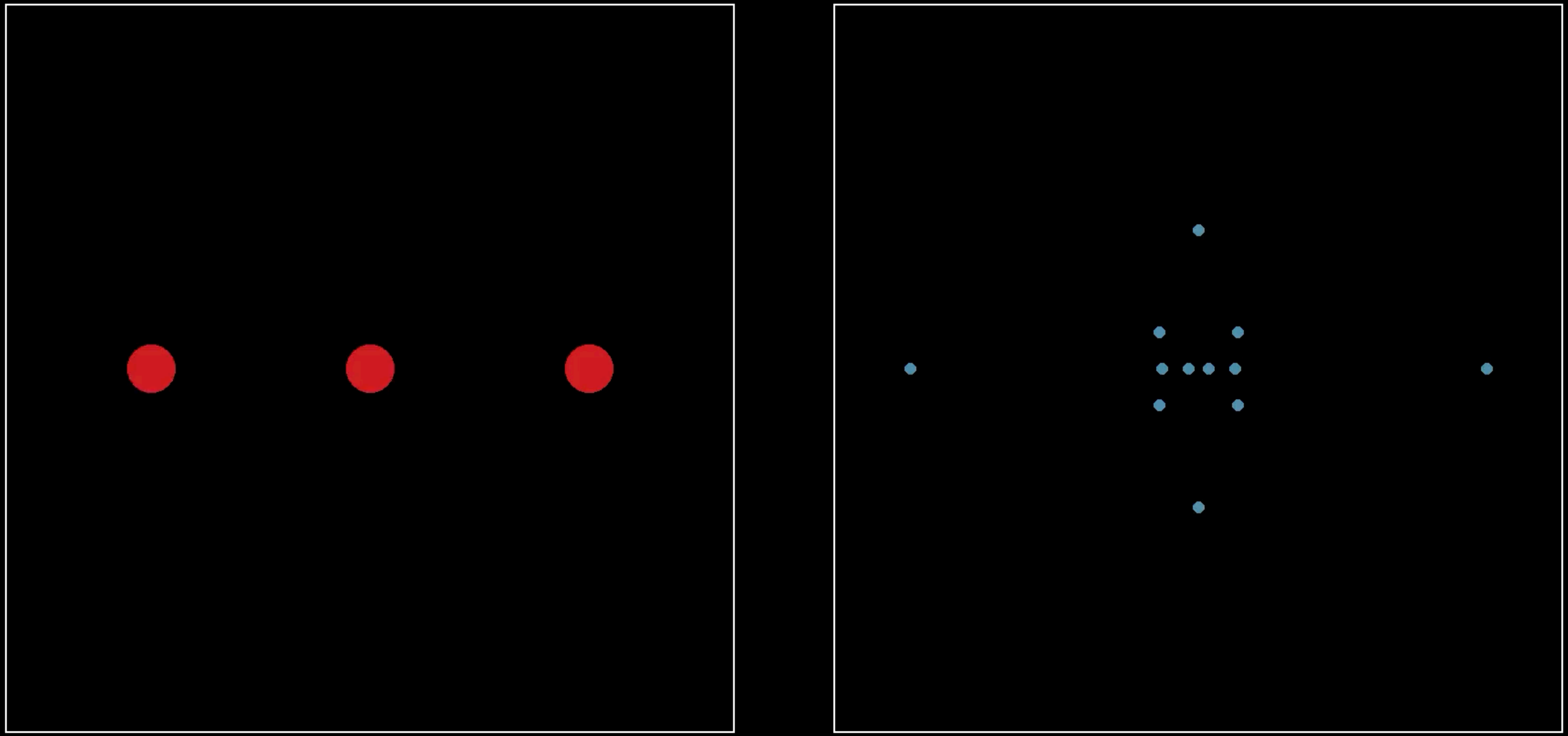

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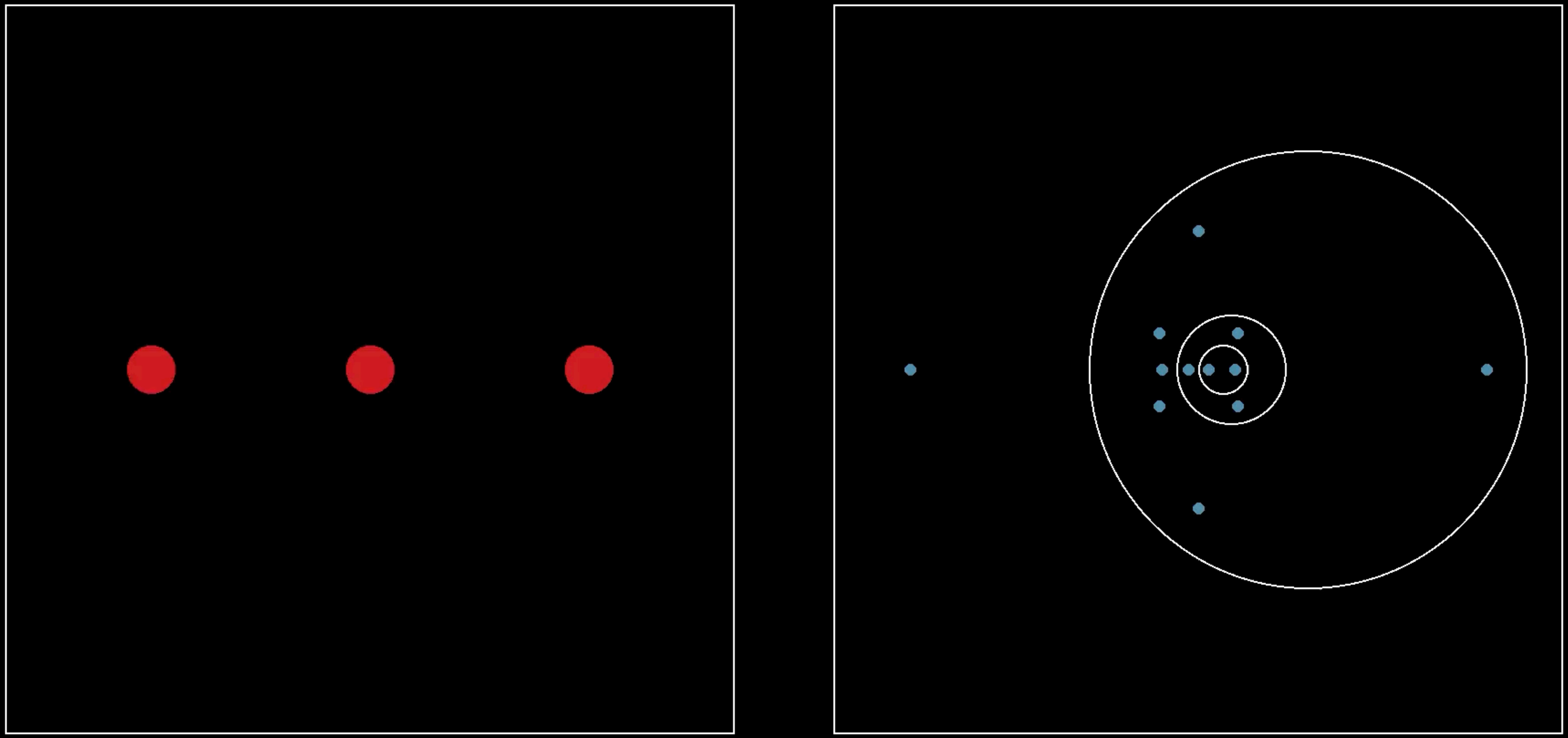

General elliptic curve construction:

For $1 < k_1 < \cdots < k_\ell$, define $\Psi_{k_1,\ldots,k_\ell}\colon \mathrm{UConf}_3\mathbb{C} \to \mathrm{UConf}_{m_{k_1}+\cdots+m_{k_\ell}}\mathbb{C}$ by

$$\Psi_{k_1,\ldots,k_\ell}(X) := \Psi_{k_1}(X) \cup \cdots \cup \Psi_{k_\ell}(X)$$

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Open problem:

Is every holomorphic map between unordered configuration spaces of points in $\mathbb{C}$ equivalent to one of the following types?

- Constant map
- Identity map
- $\Psi_{k_1,\ldots,k_\ell}$ or $R \circ \Psi_{k_1,\ldots,k_\ell}$ for some $1 < k_1 < \cdots < k_\ell$.

Thank you!