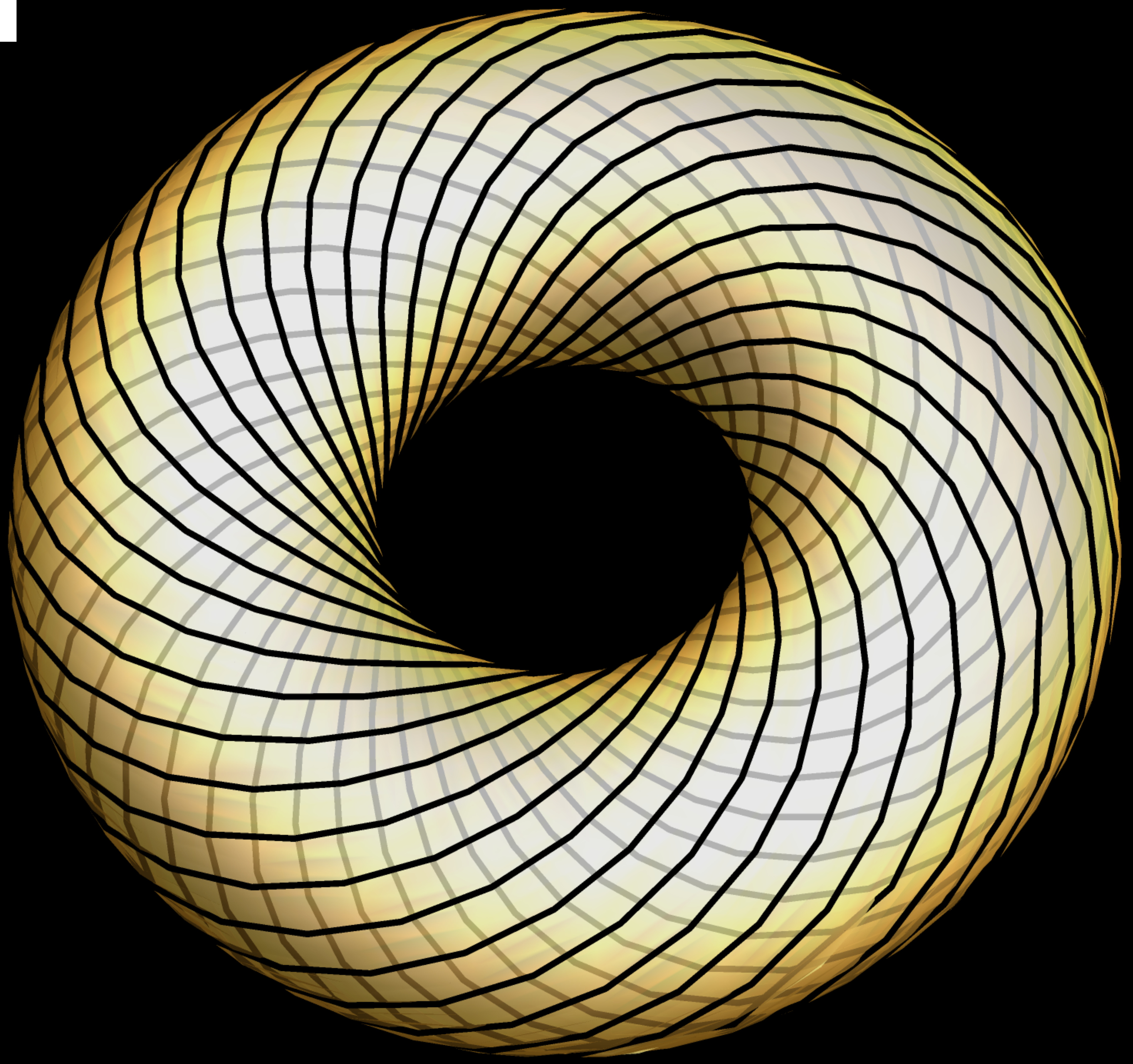
A Brief Overview of Measure Rigidity

Rose Elliott Smith
Rice University



Setup

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$f : M \to M$ is a measurable map.

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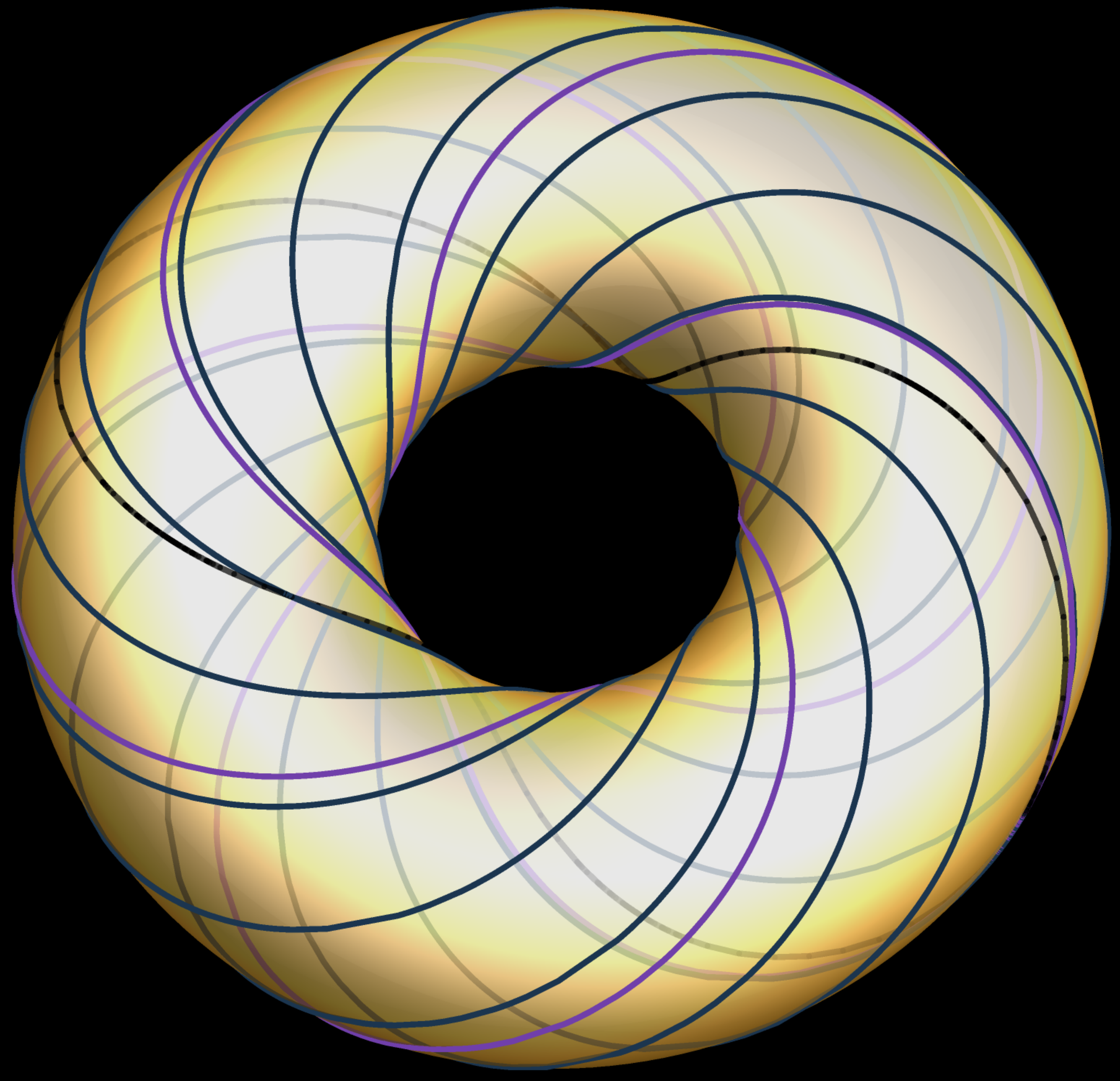
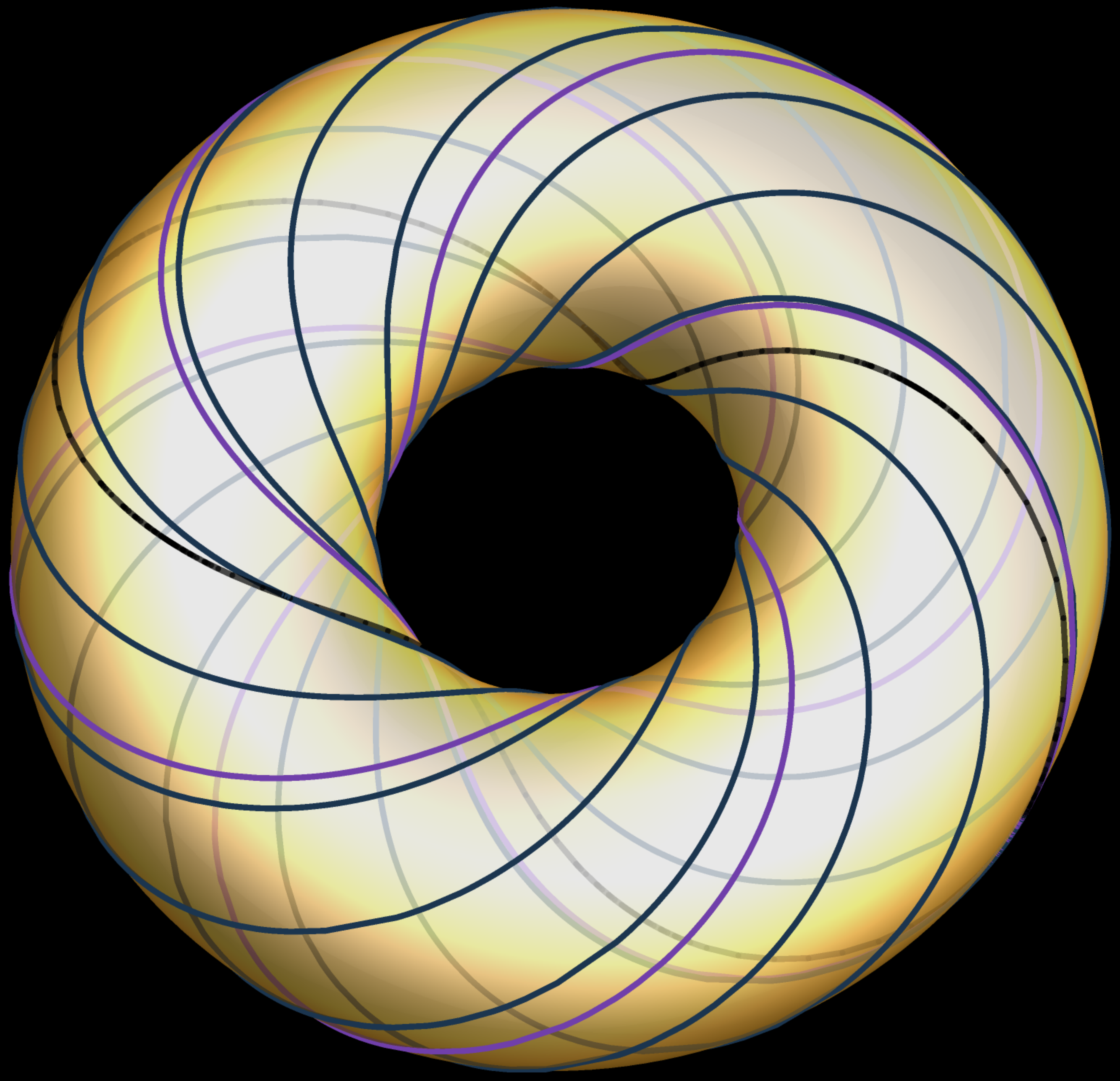
What are the orbits of f?



Motivation

$$x, f(x), f^2(x), f^3(x), \cdots$$



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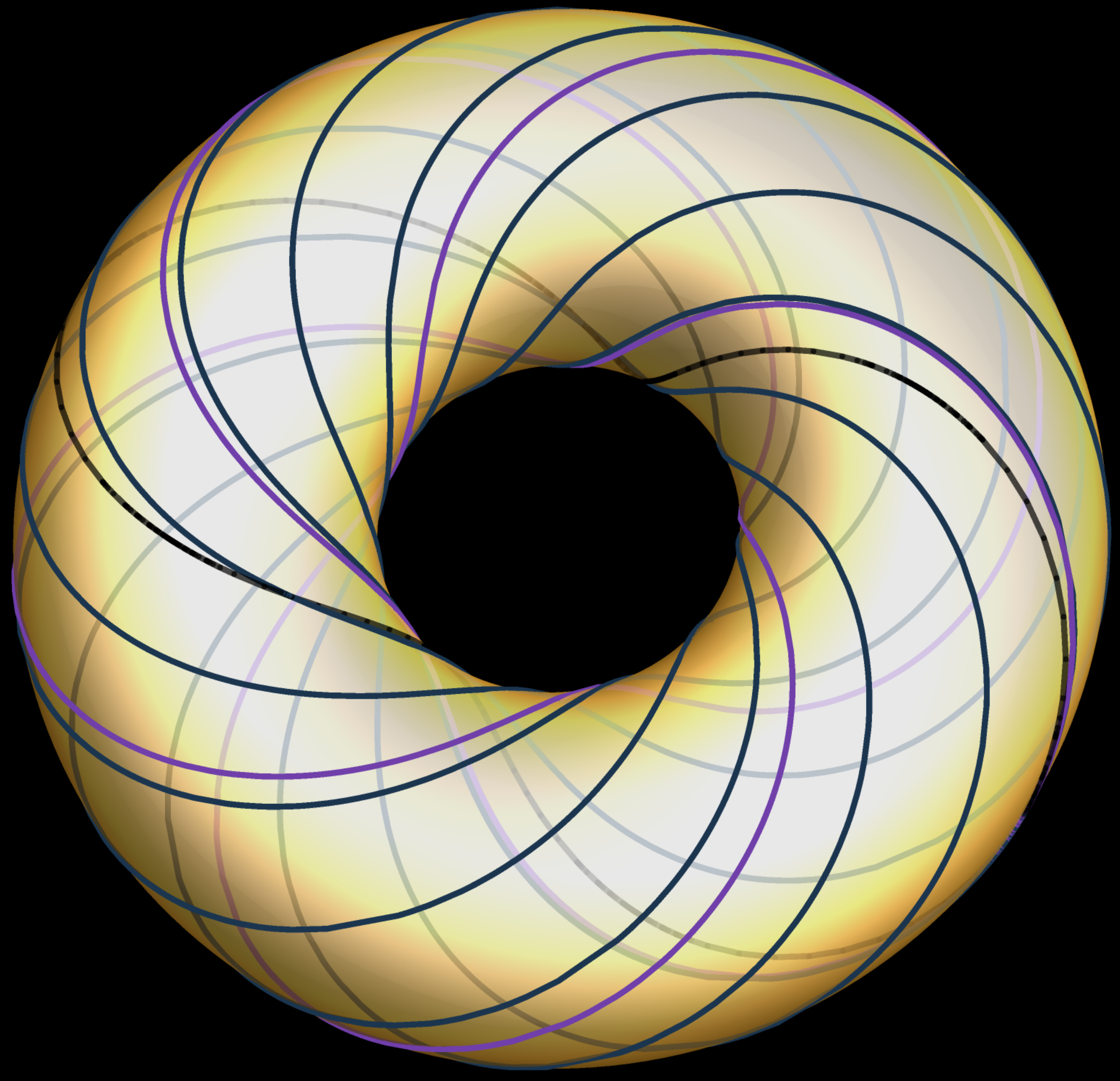

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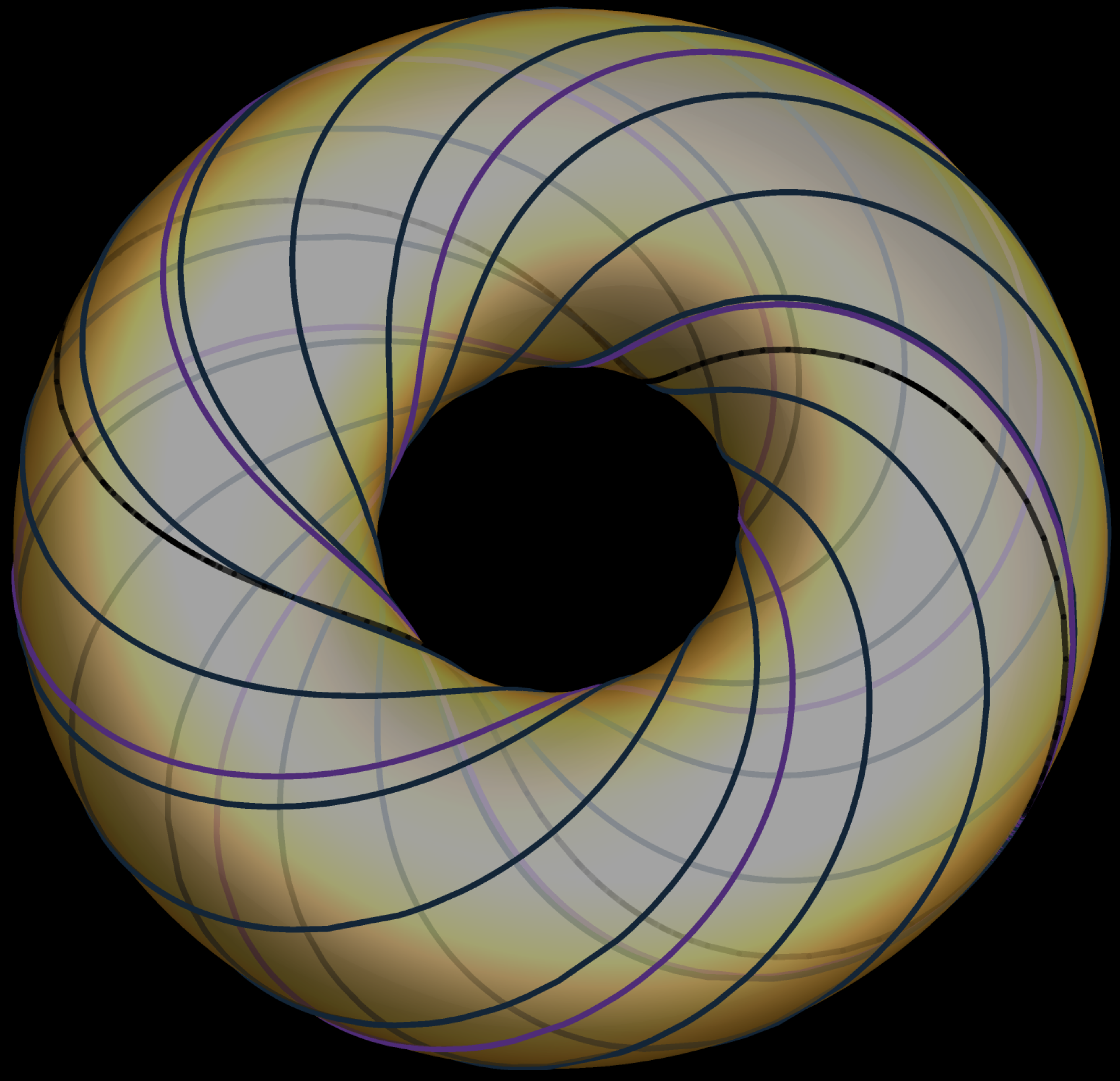

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any weak-* limit of μ_n is f-invariant

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- Teichmuller dynamics
- Complex dynamics

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What is measure rigidity?

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Statements like:
All non-atomic invariant measures are volume.
All invariant measures are finitely supported.
There is only one invariant measure.

When can you classify measures?

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This is a well-studied question. To name a few answers:
Ratner, Benoist-Quint, Eskin-Mirzakhani, Eskin-Lindenstrauss, Brown-Rodriguez Hertz, Brown-Eskin-Filip-Rodriguez Hertz…

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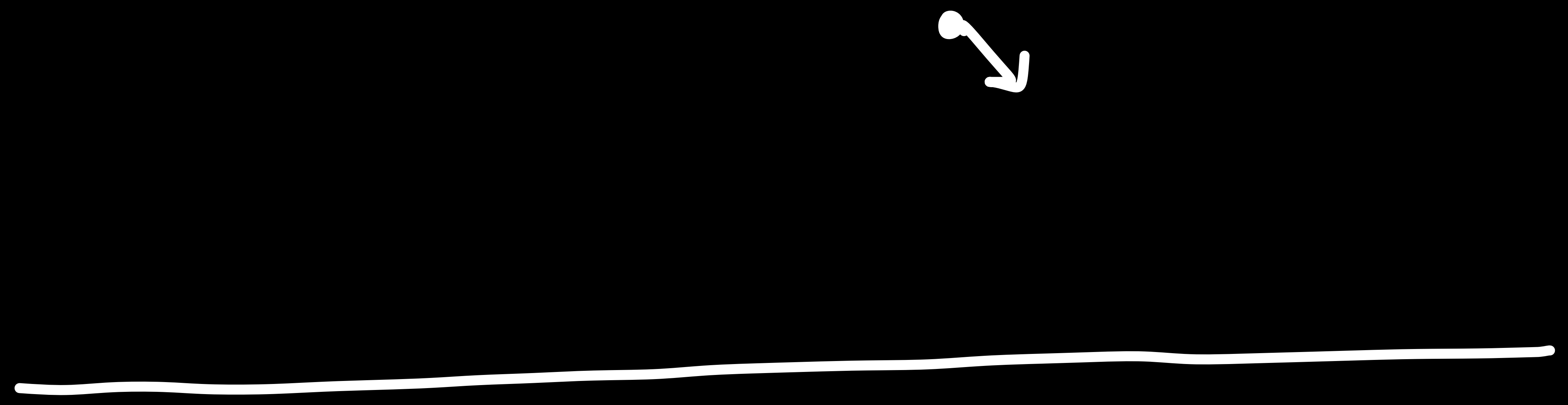

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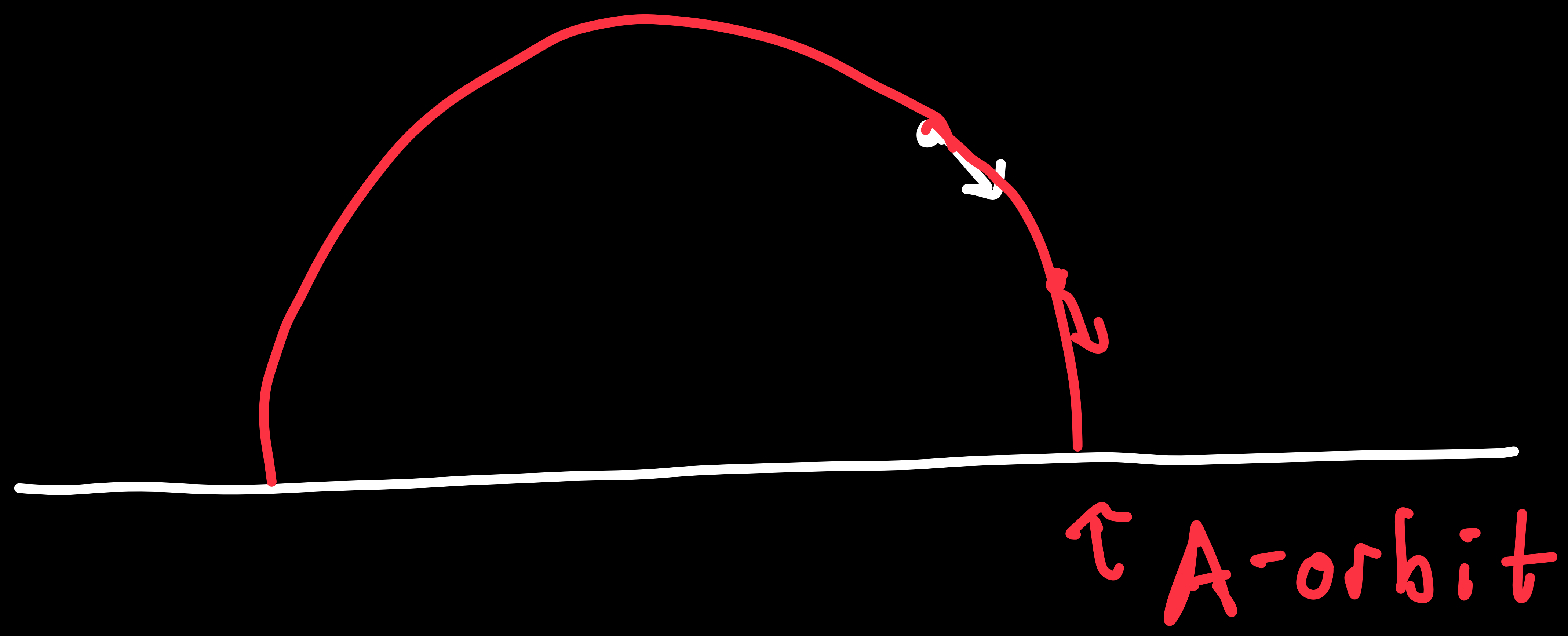

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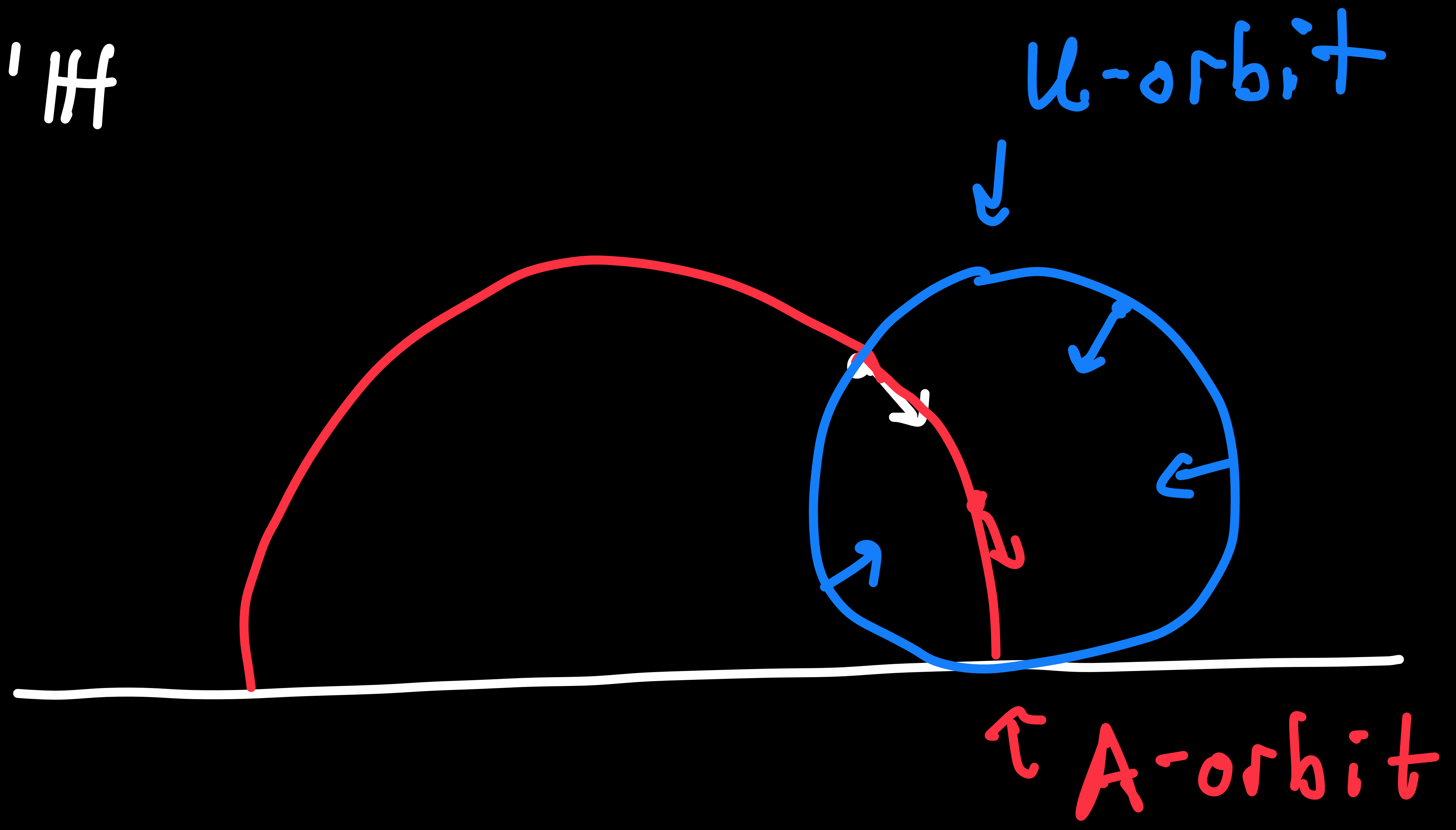

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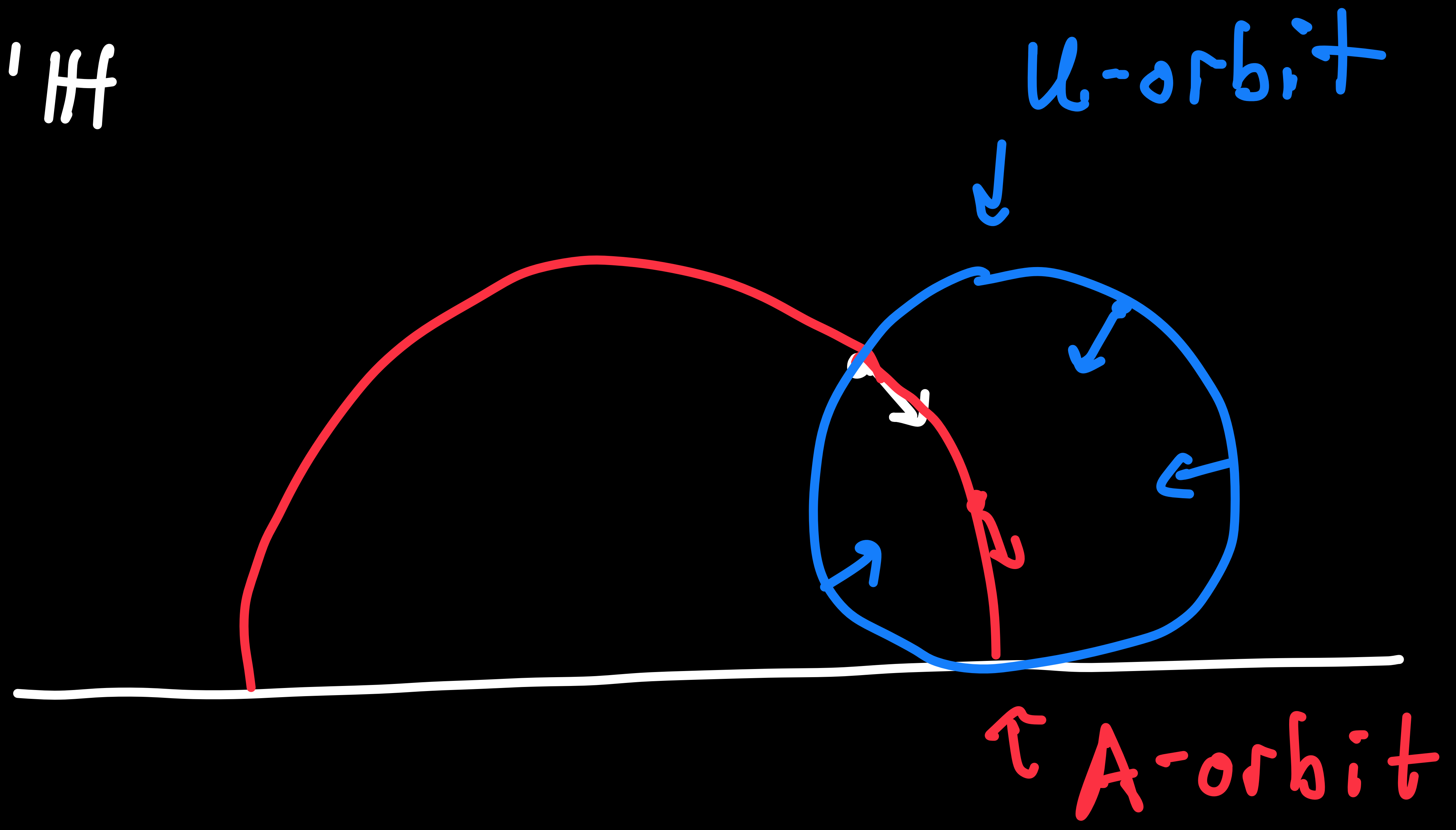

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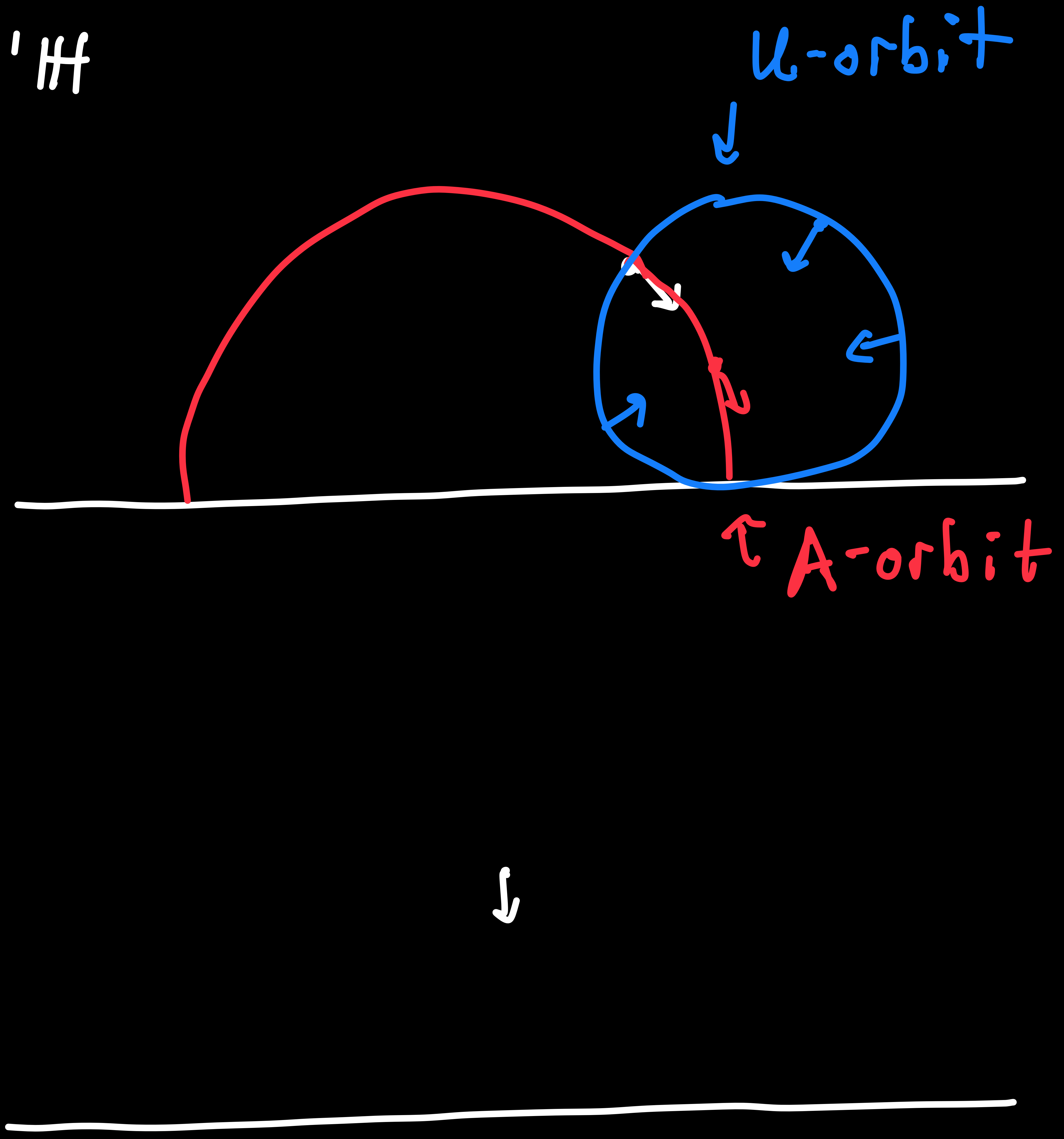

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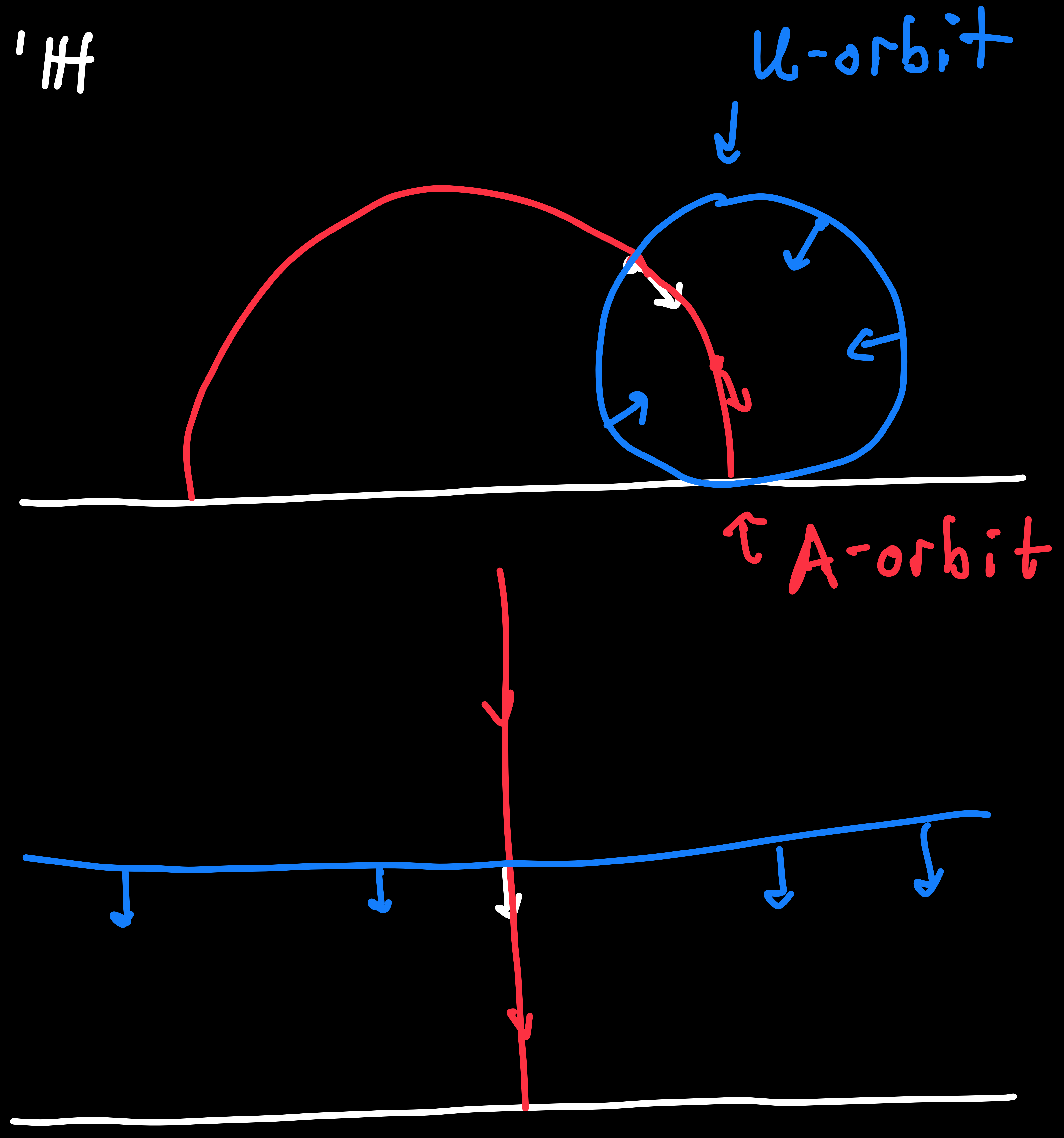

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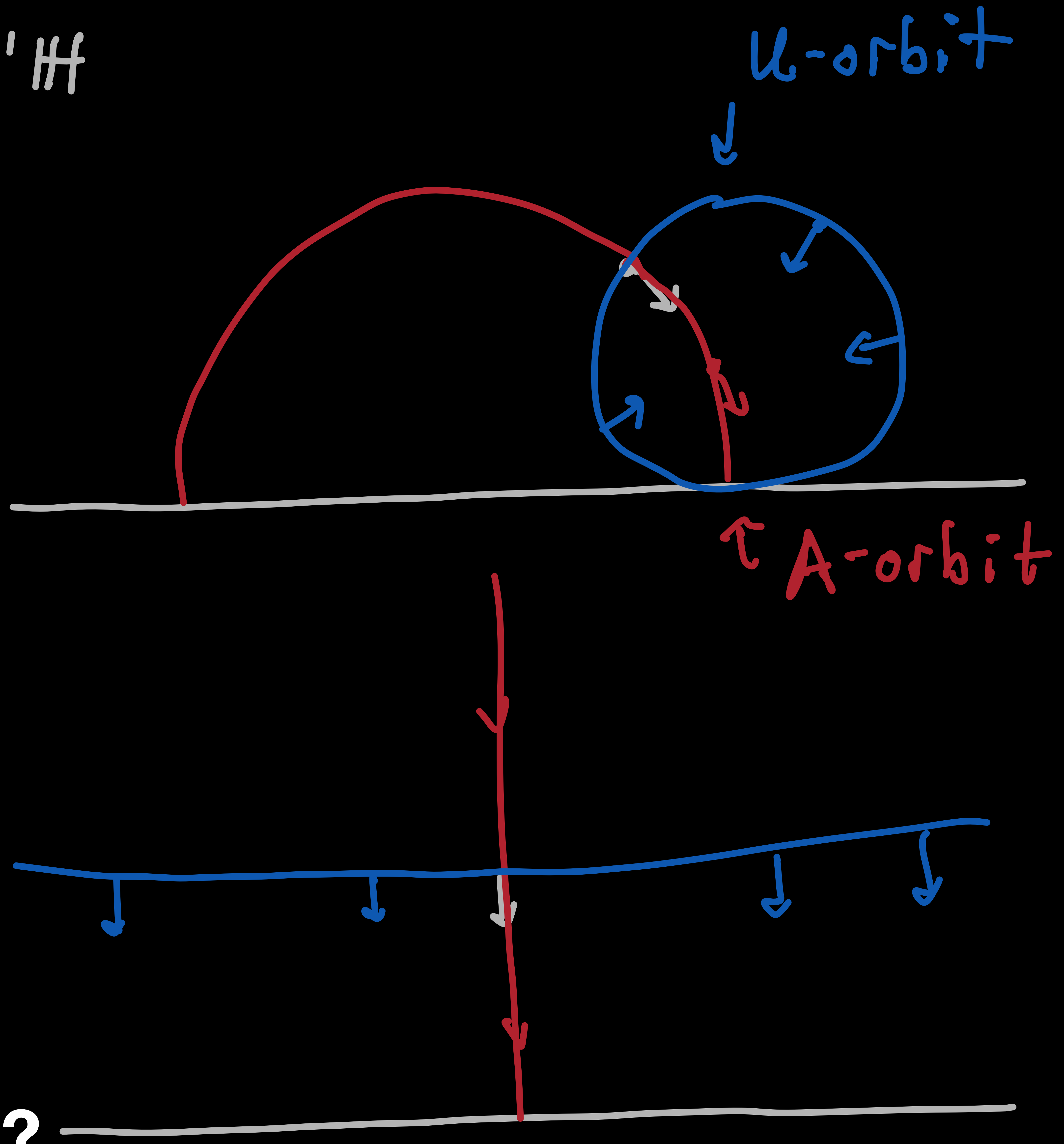
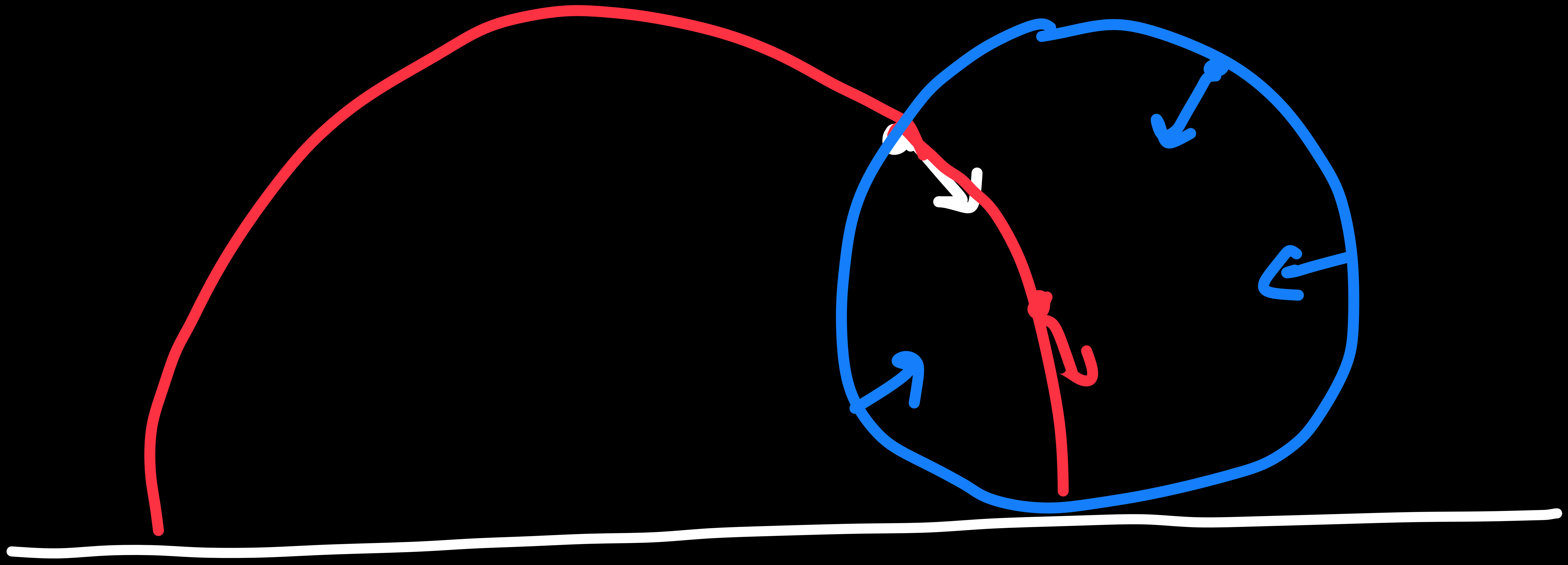


What does this look like in the quotient?

$SL_2\mathbb{R}/SL_2\mathbb{Z}$



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The image of a closed A-orbit projected onto the fundamental domain of the $SL_2\mathbb{Z}$ action on $\mathbb{H}$.

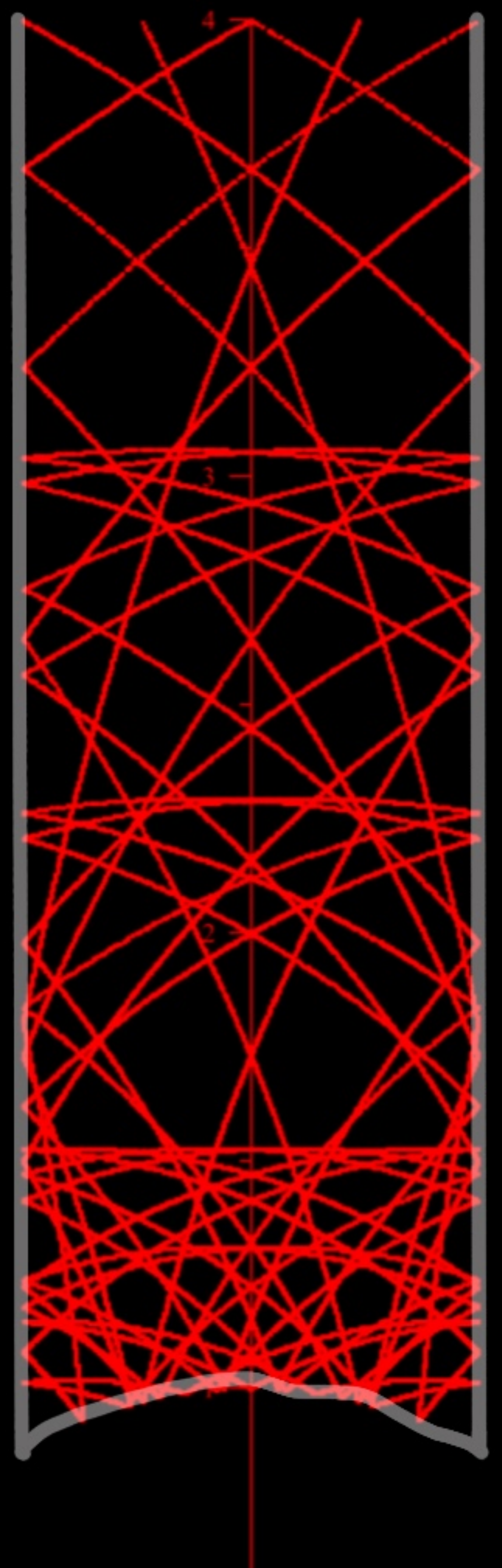
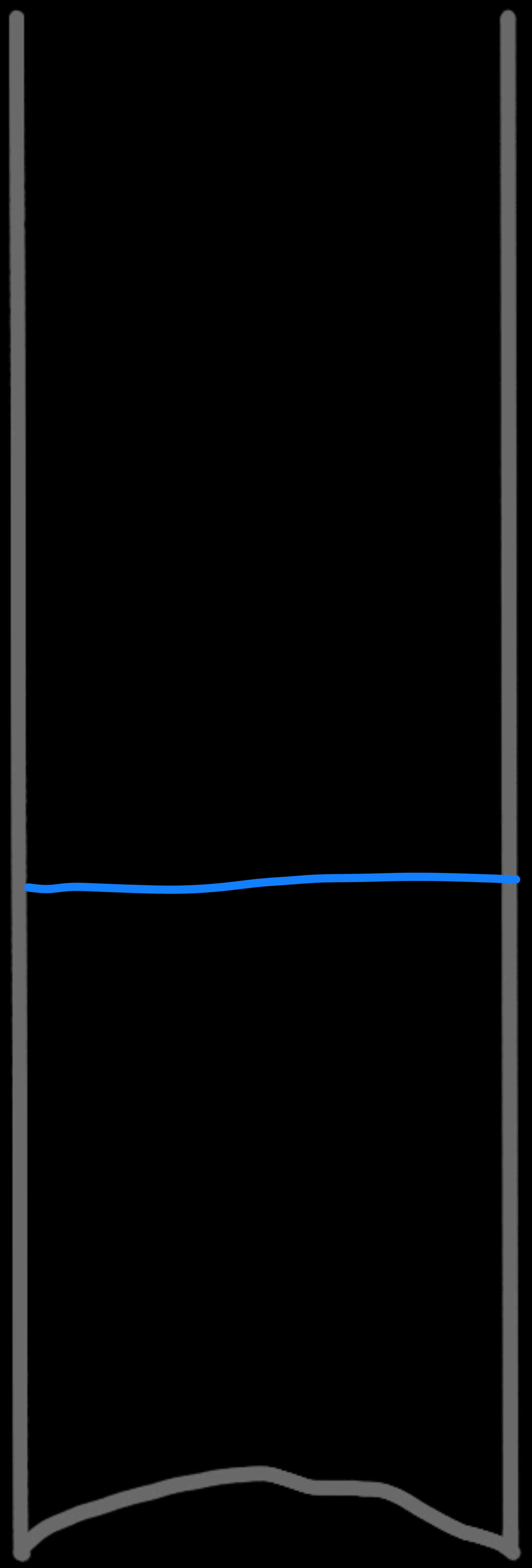


Image from "Distribution of closed geodesics on the modular surface, and Duke's theorem" Einsiedler-Lindenstrauss-Michel-Venkatesh

$SL_2\mathbb{R}/SL_2\mathbb{Z}$

The image of a closed U-orbit projected onto the fundamental domain of the $SL_2\mathbb{Z}$ action on $\mathbb{H}$.



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$\Gamma \subset G$: a lattice

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Ratner + more work

$\implies$ all orbit closures are the orbits of closed subgroups, i.e., for all $x \in M$, there is a closed subgroup $U \subset F \subset G$ such that $\overline{Ux} = Fx$.

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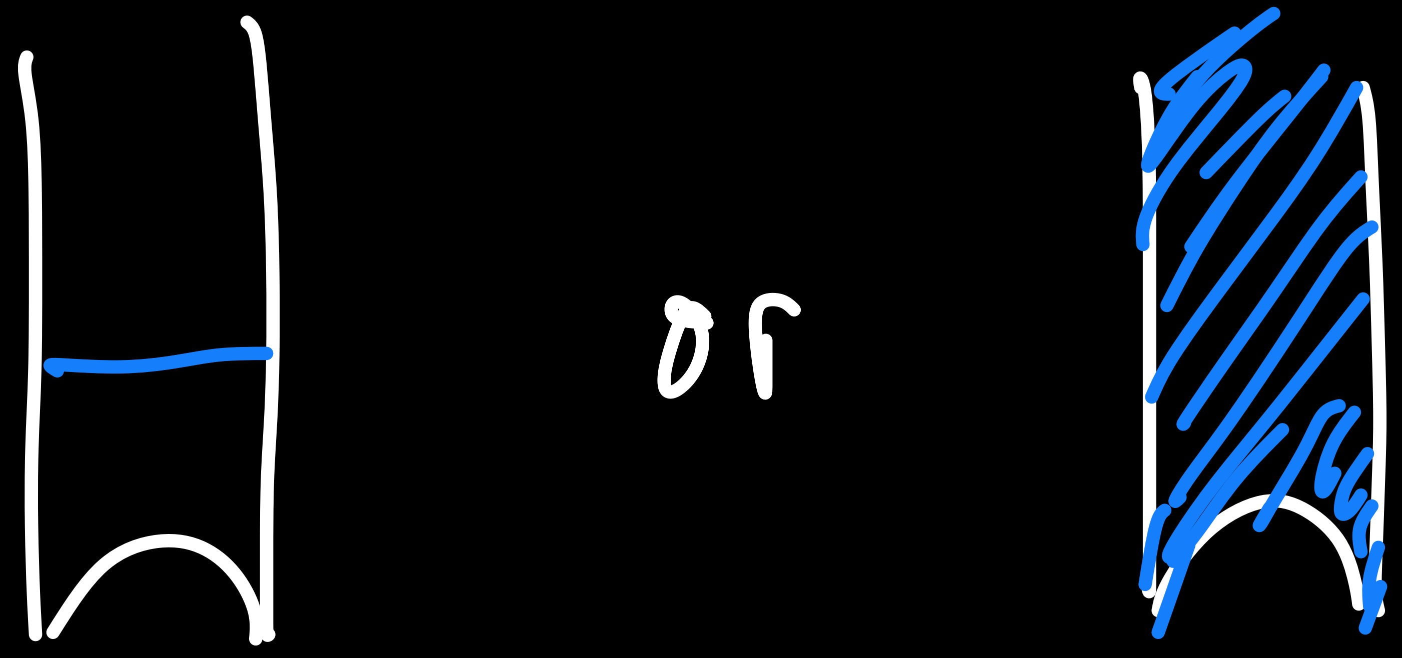

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Ratner doesn't hold— there are fractal orbit closures.

Ratner's Measure Classification Theorem

Margulis used Ratner to prove the Oppenheim Conjecture:

If $n \geq 3$ and $Q(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$ is an indefinite non-degenerate quadratic form not proportional to a rational quadratic form, then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

When can you classify measures?

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The moral: find extra invariance.

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	Assumptions	Conclusion	Setting	Year
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magic wand

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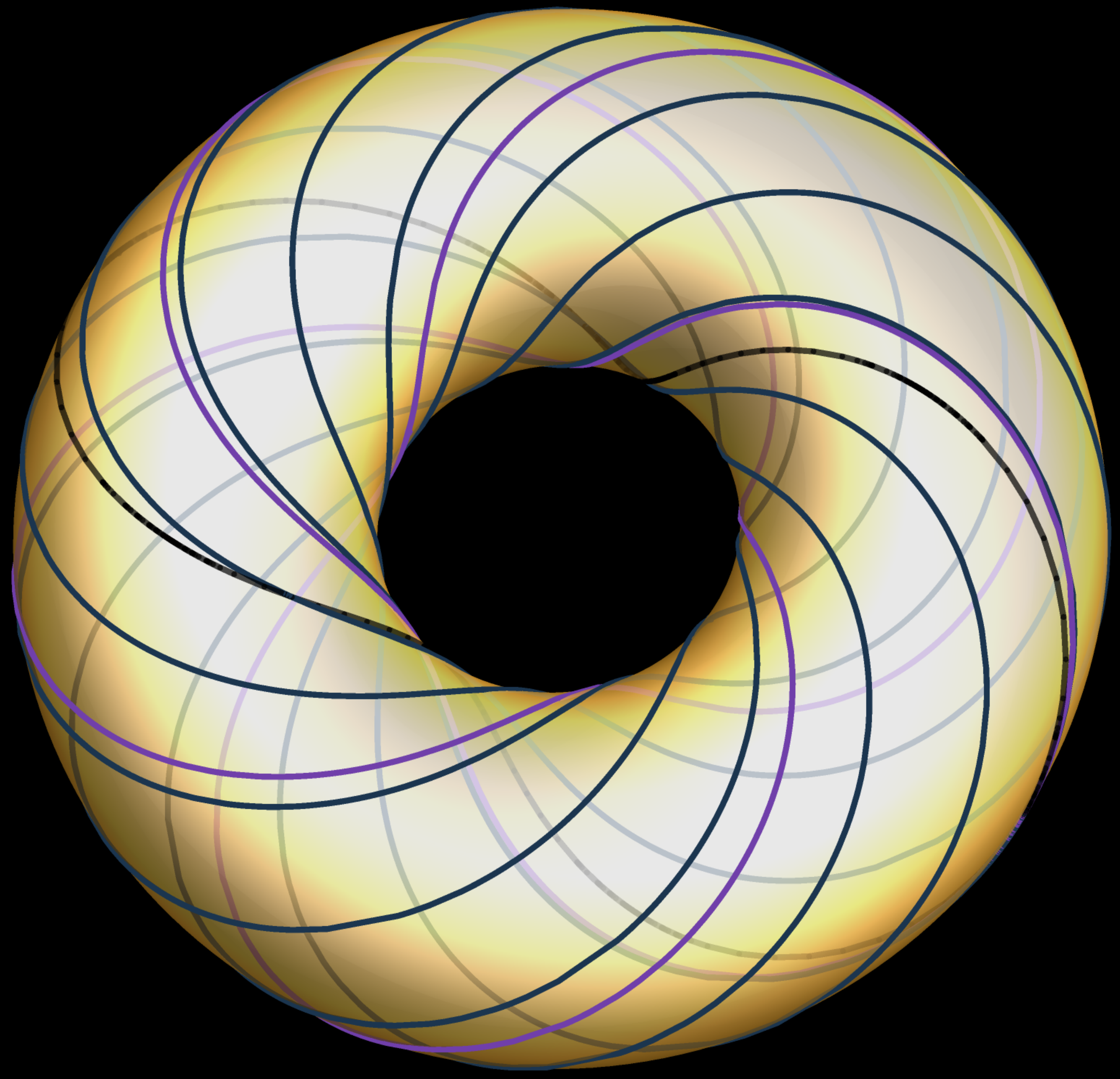
E.S. '23 ⇒ exist in a "generic" way on any manifold.



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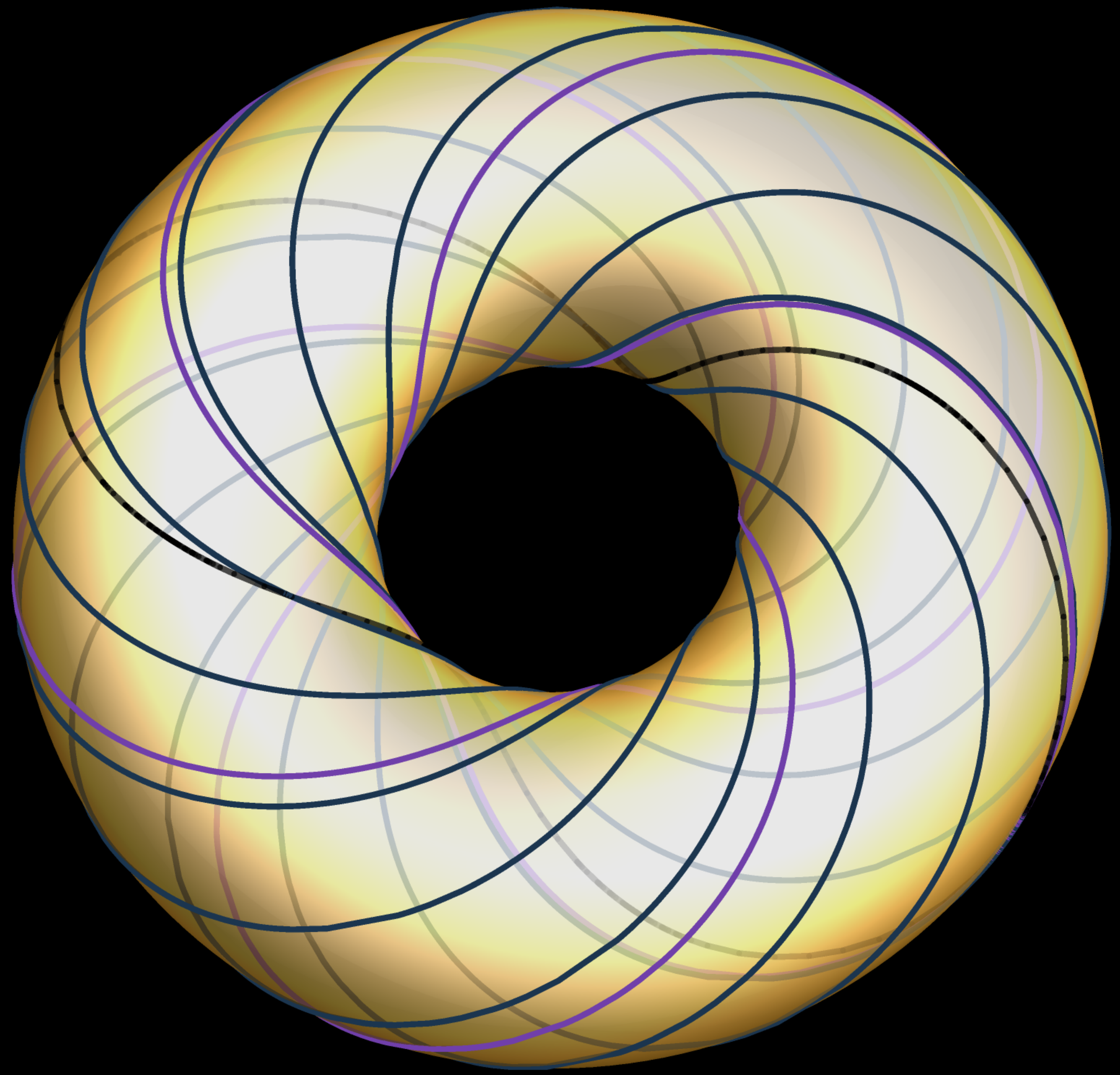
- Pesin theory
- Entropy
- Normal forms
- Exponential drift
- …



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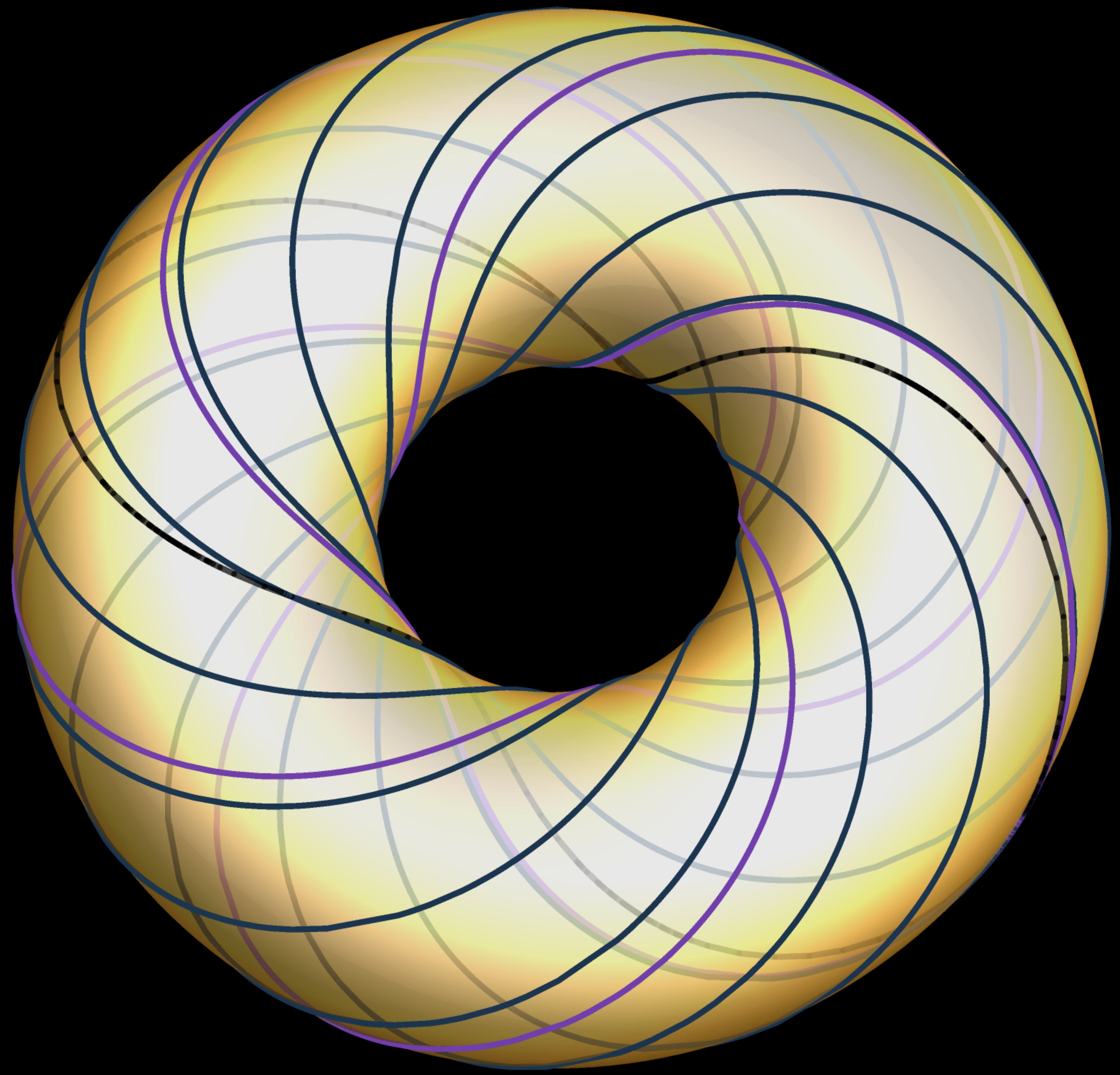

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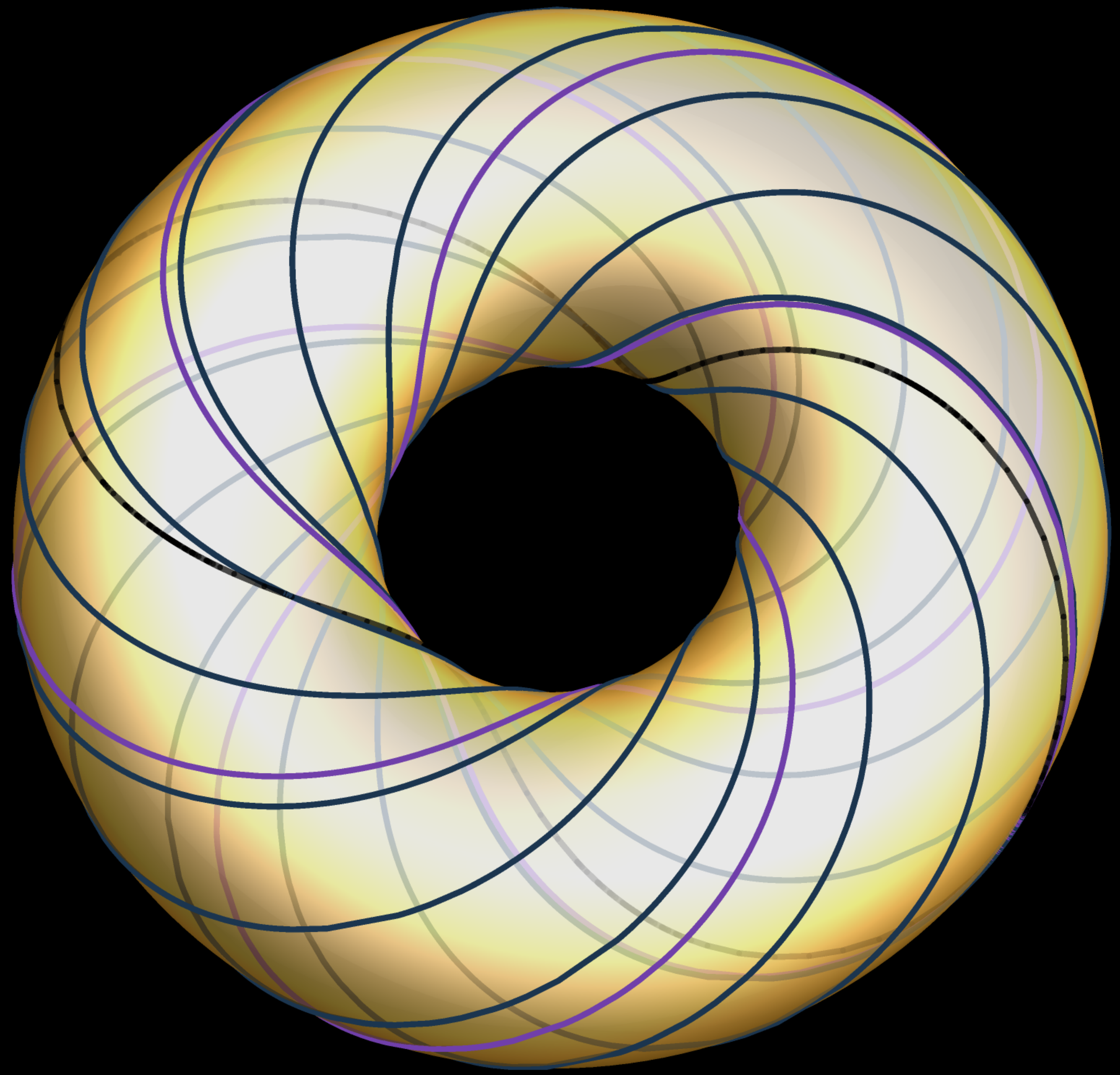

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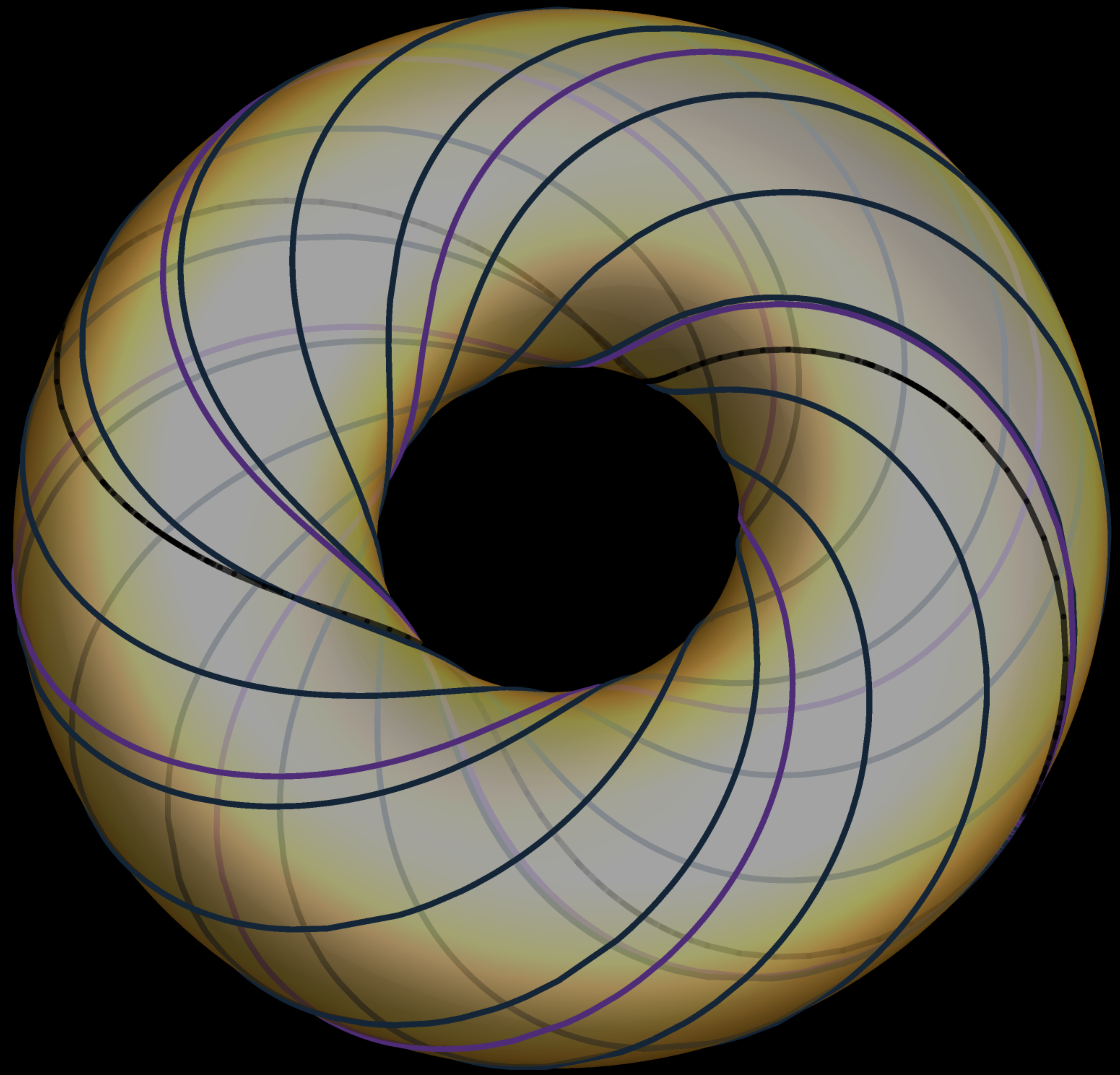

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Next time: using all of this to find physical measures

Thank you!