Harmonic Analysis and Dispersive PDE

Part I

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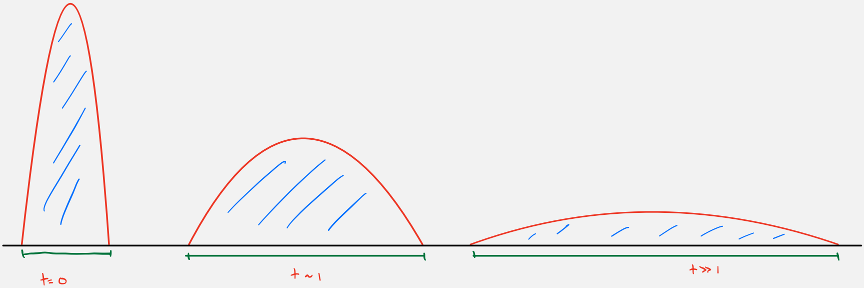
What is a dispersive PDE?

A PDE is called *dispersive* if different frequencies travel with different velocities.

Key consequence: solutions spread out or disperse as time evolves:



- Energy/Mass is conserved (so blue area stays constant),
- Solution decays pointwise (so height of red curve decreases),

$\Longrightarrow$ solution spreads out.

What is a dispersive PDE?

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A typical example is the Schrödinger equation on $\mathbb{R} \times \mathbb{R}^d$

$$i\partial_t u + \Delta u = 0, \qquad u(0) = f.$$

Notation

- Solution $u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$.
- Often suppress the spatial variable $x \in \mathbb{R}^d$ and just write $u(t)$.
- Here ∂_t is short for $\frac{\partial}{\partial_t}$ and $\Delta = \sum_{j=1}^d \partial_{x_j}^2$ is the Laplacian on $\mathbb{R}^d$.

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- Take data $f(x) = e^{ix\cdot\xi_0}$ oscillating at frequency $\xi_0 \in \mathbb{R}^d$, then solution

$$u(t,x) = e^{i(x+t\xi_0)\cdot\xi_0}$$

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  has velocity ξ_0.
- In contrast to transport equations

$$\partial_t u + v \cdot \nabla u = 0 \qquad \text{(all frequencies have same velocity)}$$

  and parabolic PDE

$$-\partial_t u + \Delta u = 0 \qquad \text{(all frequencies decay to zero).}$$

Norms

Some additional notation:

- For a function $f : \mathbb{R}^d \to \mathbb{C}$ we take

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}^d} |f(x)|^p dx \Big)^{\frac{1}{p}}, \qquad \|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|.$$

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- Also require a space measure smoothness, here $\nabla = (\partial_1, \dots, \partial_d)$,

$$\|f\|_{H^1} = \Big(\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 \Big)^{\frac{1}{2}}, \qquad \|f\|_{\dot{H}^1} = \|\nabla f\|_{L^2}.$$

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- Given a space-time map $u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ we take

$$\|u\|_{L^q_t L^r_x} = \Big\| \|u(t,x)\|_{L^r_x} \Big\|_{L^q_t}, \qquad \|u\|_{L^\infty_t H^1_x} = \Big\| \|u(t,x)\|_{H^1_x} \Big\|_{L^\infty_t}.$$

The linear problem

Given data f, find a (unique) solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ such that

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Can solve linear problem explicitly!

Assume that f decays, then have the unique global solution

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)dy.$$

Consequences

Solution to linear Schrödinger equation given by

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)\,dy$$

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- Clearly solution decays pointwise as $t \to \infty$. In fact have dispersive bound

$$\|u(t)\|_{L^\infty_x(\mathbb{R}^d)} \leqslant |t|^{-\frac{d}{2}} \|f\|_{L^1}.$$

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$$\|u(t)\|_{L^\infty_x(\mathbb{R}^d)} \leqslant |t|^{-\frac{d}{2}} \|f\|_{L^1}.$$

- By differentiating in time, and integrating by parts, can also conclude the conservation of mass and energy

$$\|u(t)\|_{L^2_x} = \|f\|_{L^2}, \qquad \|\nabla u(t)\|_{L^2_x} = \|\nabla f\|_{L^2}.$$

Thus solution disperses as $t \to \infty$.

Issues

Solution to linear Schrödinger equation given by

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)dy$$

Not at all obvious that $u(t) \to f$ as $t \to 0$!

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- Convergence in norm (say H^1 or L^2) true.
- Problem of pointwise convergence, namely

$$\lim_{t\to\infty} u(t,x) = f(x)$$

dates back to Carleson1979. True for smooth data $f \in H^1$ but can in fact fail for rough data $f \in L^2$ Dahlberg-Kenig1982, Bourgain2016 (sharp regularity required still open question).

Issues

Solution to linear Schrödinger equation given by

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)dy$$

- Smooth data can lead to singular solutions

$$u(t,x) = \frac{1}{(4\pi i(t-1))^{\frac{d}{2}}} e^{i\frac{|x|^2}{4(t-1)}}.$$

- Loss of uniqueness: there exists (smooth) solutions $u(t,x)$ s.t. $u = 0$ for $t \leqslant 0$ and $u \neq 0$ for $t > 0$! Schrödinger analogue of Tychonoff's construction for the heat equation.

Both (1) + (2) occur as examples have no decay as $|x| \to \infty$.

Energy can disperse from $\infty \implies$ counteracts dispersive decay.

The nonlinear problem

Given data f, find a (unique) solution $u : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$ such that

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

- Fixed $d = 4$ as numerology works out nicely. This is known as the focusing energy critical nonlinear Schrödinger equation (NLS).

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- Problem heavily studied in all dimensions $d \geqslant 1$ Strichartz1977, Cazenave-Weissler1990, Bourgain1999, Colliander-Keel-Staffilani-Takaoka-Tao2004, Kenig-Merle2006, Dodson2019, ...

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Immediate Issue:
In contrast to linear case, no formula for solution!

$\Longrightarrow$ need to use alternative arguments to study dynamics.

Linear vs Nonlinear behaviour

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

Roughly speaking, the evolution of a nonlinear dispersive PDE is a combination of three regimes:

1. Linear terms dominate.
2. Nonlinear interactions dominate.
3. Intermediate (or critical) behaviour.

Linear terms dominate

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

- Solution exists globally in time, and converges to a linear solution at $t = \infty$ (*scattering*).
- Essentially all global bounds satisfied by the free evolution, also hold for the nonlinear evolution.
- Nonlinear effects are weak, and can only occur for short times.
- Typically this scenario tends to hold for small data, short times, high regularities, ...

Nonlinear interactions dominate

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

- Solutions can form singularities in finite times (or even instantaneously), thus even local well-posedness can fail. If conserved quantities prevent blow-up (i.e. non-focusing nonlinearities), then solution is unstable for short to medium times.
- Typically this behaviour arises for large data, low regularities, long times, strong nonlinearities ...

Nonlinear interactions dominate

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

Theorem (Glassey1977)

Assume $f \in H^1$ satisfies $\||x|f\|_{L^2} < \infty$ and assume energy is negative

$$\mathcal{E}_{NLS}[f] = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla f|^2 - \frac{1}{4}|f|^4 dx < 0.$$

Then solution u forms singularity in finite time.

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Key idea in proof is to compute the second derivative in time of the virial quantity

$$V(t) = \int_{\mathbb{R}^4} |x|^2 |u(t,x)|^2 dx.$$

This is negative(!) so $V(t)$ concave $\Rightarrow V(t)$ must reach zero in finite time.

Intermediate or critical behaviour

- The linear and nonlinear effects are balanced. This leads to modifications to the asymptotic behaviour (i.e. log corrections to the expected linear behaviour) or non-dispersive solitary wave, or soliton like solutions.

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- The KdV equation

$$\partial_t u + \partial_{xxx} u + u \partial_x u = 0$$

has the (explicit) soliton solutions

$$u(t, x) = Q(x - t),$$

where $Q(x) = \frac{1}{2}\text{sech}^2(\frac{x}{2})$.

- First observed by Scott Russell in 1834 in a canal in Scotland.

Intermediate or critical behaviour

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

Focusing NLS has similar explicit solutions:

$$Q(x) = \frac{8}{8 + |x|^2}$$

- The ground state Q (or Aubine-Talenti function `Aubin1976, Talenti1976`) can be characterised as the optimiser (up to translations and rescalings) of the Sobolev embedding

$$\|g\|_{L^4(\mathbb{R}^4)} \leqslant C_d \|\nabla g\|_{L^2(\mathbb{R}^4)}.$$

- Also satisfies the elliptic equation

$$\Delta Q = -Q^3.$$

- Clearly the solution $u(t,x) = Q(x)$ is far from linear!

The Small Data Regime

Intuition:
If data 'small', we expect solution u to be small, and so $|u|^2 u$ very small.

$$\implies u \text{ should be close to linear solution } (i\partial_t + \Delta)u_L = 0.$$

What do we mean by small?

A good choice is to use norm

$$\|f\|_{H^1(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |f(x)|^2 + |\nabla f(x)|^2 dx \Big)^{\frac{1}{2}}$$

Why is this a good choice?

Corresponds to 'energy' of solution, plays key role in controlling large time dynamics in $4d$.

The nonlinear problem: Small data global existence

Theorem (Cazenave-Weissler1990)

There exists $\epsilon > 0$ such that for any data $\|f\|_{H^1(\mathbb{R}^4)} < \epsilon$ there exists a (unique) solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^4))$ to

$$(i\partial_t + \Delta)u = -|u|^2 u, \qquad u(0) = f.$$

Moreover, there exists linear solutions $(i\partial_t + \Delta)u_{\pm\infty} = 0$ such that

$$\lim_{t\to\pm\infty} \|u(t) - u_{\pm\infty}(t)\|_{H^1(\mathbb{R}^4)} = 0.$$

- This is a small data result as requires that the data $f \in H^1(\mathbb{R}^4)$ satisfies

$$\|f\|_{H^1(\mathbb{R}^4)} < \epsilon.$$

- The space $C(\mathbb{R}; H^1(\mathbb{R}^4))$ is the collection of all continuous functions $u(t) : \mathbb{R} \to H^1(\mathbb{R}^4)$.

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Moreover, there exists linear solutions $(i\partial_t + \Delta)u_{\pm\infty} = 0$ such that

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- The final property is called scattering. It states that for large times the solution $u(t)$ converges to a solution to the linear equation.

$\implies$ Linear terms dominate dynamics in the small data case.

What is a solution?

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

First issue: Data is in H^1, so at most can expect u to also be H^1. What does it mean for u to be a solution?

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- Define the Fourier transform

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^4} e^{-ix\cdot\xi} f(x)dx$$

  and let $e^{it\Delta}$ denote the free propagator

$$e^{it\Delta} f = \mathcal{F}^{-1}[e^{-it|\xi|^2}\widehat{f}(\xi)](x).$$

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- Applying Duhamel formula/variation of constants, we can write solution in the integral form

$$u(t) = e^{it\Delta} f - i \int_0^t e^{i(t-s)\Delta}(|u|^2 u)(s)ds.$$

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- So by solution, we really mean a solution to the integral formulation of the equation.

Ideas in Proof

Define

$$\Phi(u) = e^{it\Delta} f - i \int_0^t e^{i(t-s)\Delta}(|u|^2 u)(s) ds$$

Goal: Find a nice Banach space $X \subset C(\mathbb{R}; H^1)$ such that

1. We have the linear bound

$$\|e^{it\Delta} f\|_X \lesssim \|f\|_{H^1}.$$

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Standard fixed point argument (i.e. Picard existence theorem from ODE) then implies that Φ has a (unique) fixed point in a small ball around $e^{it\Delta} f \in X$.

$\implies$ get solution to integral formulation of problem

Linear Estimates

Theorem (Strichartz1977)

We have

$$\|e^{it\Delta}f\|_{L^3_{t,x}\cap L^\infty_t L^2_x} \lesssim \|f\|_{L^2}, \qquad \Big\| \int_0^t e^{i(t-s)\Delta}F(s)ds\Big\|_{L^3_{t,x}\cap L^\infty_t L^2_x} \lesssim \|F\|_{L^{\frac{3}{2}}_{t,x}}.$$

- Start of a long story connecting problems in harmonic analysis to estimates for dispersive PDE. Estimates of these type go via the name Strichartz Estimates. At least on $\mathbb{R}^n$, optimal result due to Keel-Tao1998.

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- Start of a long story connecting problems in harmonic analysis to estimates for dispersive PDE. Estimates of these type go via the name Strichartz Estimates. At least on $\mathbb{R}^n$, optimal result due to Keel-Tao1998.
- Above suggests we should take our solution space X to be

$$\|u\|_X = \|u\|_{L^\infty_t H^1_x} + \|\nabla u\|_{L^3_{t,x}}.$$

  The above Strichartz estimate then gives

$$\|\Phi(u)\|_X = \Big\|e^{it\Delta}f - i\int_0^t e^{i(t-s)\Delta}(|u|^2u)(s)ds\Big\|_X \lesssim \|f\|_{H^1} + \|\nabla(|u|^2u)\|_{L^{\frac{3}{2}}_{t,x}}.$$

Nonlinear Estimate

Only remains to prove nonlinear estimate

$$\|\nabla(|u|^2 u)\|_{L_x^{\frac{3}{2}}} \lesssim \|u\|_{L^3_{t,x}}^2 \|u\|_{L_t^\infty H^1}$$

but this is an easy consequence of Hölder's inequality: if $\frac{1}{r} = \frac{1}{a} + \frac{1}{b}$ then

$$\|fg\|_{L^r} \leqslant \|f\|_{L^a} \|g\|_{L^b},$$

together with the Sobolev embedding

$$\|f\|_{L_x^{12}(\mathbb{R}^4)} \lesssim \|\nabla f\|_{L_x^3(\mathbb{R}^4)}.$$

Summary

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

- Small data theory in H^1 for the nonlinear problem is essentially a consequence of the linear Strichartz estimate

$$\|e^{it\Delta} f\|_{L^3_{t,x}} \lesssim \|f\|_{L^2}.$$

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- This estimate can be thought of as another way to capture the dispersive properties of the evolution, that very conveniently only requires $f \in L^2$, as opposed to the usual dispersive estimate

$$\|e^{it\Delta} f\|_{L^\infty_x} \lesssim |t|^{-2} \|f\|_{L^1_x}.$$

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- Essentially the same strategy underlies much of the progress in dispersive PDE over the past two decades, at least in the small data or linear regime.

The nonlinear problem II: nonperturbative setting

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

What happens for large data?

Have some hope to control dynamics as the energy

$$\mathcal{E}_{NLS}[f] = \int_{\mathbb{R}^4} \frac{1}{2}|\nabla f|^2 - \frac{1}{4}|f|^4 dx$$

and mass

$$\mathcal{M}[f] = \int_{\mathbb{R}^4} |f|^2 dx$$

are both conserved under the nonlinear flow

$$\mathcal{E}_{NLS}[u(t)] = \mathcal{E}_{NLS}[u(0)], \qquad \mathcal{M}[u(t)] = \mathcal{M}[u(0)].$$

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Bad news:

(1) Energy and mass alone not sufficient to control dynamics.

(2) Unlike small data case, nonlinear interactions can dominate!

For instance:

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For instance:

- For any sufficiently decaying data with negative energy

$$\mathcal{E}_{NLS}[f] < 0,$$

  the solution $u(t)$ forms a singularity in finite time Glassey1977.

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Bad news:

(1) Energy and mass alone not sufficient to control dynamics.

(2) Unlike small data case, nonlinear interactions can dominate!

For instance:

- Define $Q(x) = \frac{8}{8+|x|^2}$. Then $\Delta Q = -Q^3$ and hence

$$u(t,x) = Q(x)$$

  is a global (nondispersive!) solution.

The nonlinear problem II: Scattering below ground state

Theorem (Dodson2019)

Assume that $f \in H^1(\mathbb{R}^4)$ with

$$\mathcal{E}_{NLS}[f] < \mathcal{E}_{NLS}[Q] \qquad \text{and} \qquad \|f\|_{L^4(\mathbb{R}^4)} < \|Q\|_{L^4(\mathbb{R}^4)}.$$

Then there exists a unique global solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ to

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

Moreover $u(t)$ scatters to a linear solution as $t \to \pm\infty$.

Builds on many previous results:
Cazenave-Weissler1990, Bourgain1999,
Colliander-Keel-Staffilani-Takaoka-Tao2004, Kenig-Merle2006,
Killip-Visan2010, ...

The nonlinear problem II: Scattering below ground state

Theorem (Dodson2019)

Assume that $f \in H^1(\mathbb{R}^4)$ with

$$\mathcal{E}_{NLS}[f] < \mathcal{E}_{NLS}[Q] \qquad \text{and} \qquad \|f\|_{L^4(\mathbb{R}^4)} < \|Q\|_{L^4(\mathbb{R}^4)}.$$

Then there exists a unique global solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ to

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad u(0) = f.$$

Moreover $u(t)$ scatters to a linear solution as $t \to \pm\infty$.

The condition $\|f\|_{L^4(\mathbb{R}^4)} < \|Q\|_{L^4(\mathbb{R}^4)}$ is necessary to ensure global existence! In fact, if

$$f \text{ radial}, \quad \mathcal{E}_{NLS}[f] < \mathcal{E}_{NLS}[Q], \quad \text{and } \|f\|_{L^4(\mathbb{R}^4)} > \|Q\|_{L^4(\mathbb{R}^4)}$$

then solution u blows up in finite time Kenig-Merle2006.

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The condition $\|f\|_{L^4(\mathbb{R}^4)} < \|Q\|_{L^4(\mathbb{R}^4)}$ ensures that the energy is coercive, namely

$$\mathcal{E}_{NLS}[u](t) = \int_{\mathbb{R}^4} \frac{1}{2}|\nabla u(t)|^2 - \frac{1}{4}|u(t)|^4 dx \approx \|u(t)\|_{\dot{H}^1}^2.$$

Although $\|u(t)\|_{L^4}$ not conserved, the constraint $\|u(t)\|_{L^4} < \|Q\|_{L^4}$ is preserved under the nonlinear evolution.

The nonlinear problem II: Scattering below ground state

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Moreover $u(t)$ scatters to a linear solution as $t \to \pm\infty$.

The dynamics above the ground state remain an important open problem.

Closely connected to the soliton resolution conjecture which states that all global in time solutions should eventually resolve into sum of solitons together with a dispersive error.