Harmonic Analysis and Dispersive PDE

Part II

Timothy Candy

University of Otago

University of Otago
ŌTĀKOU WHAKAIHU WAKA

Department of
Mathematics and Statistics

The Fourier Restriction Problem

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} d\xi.$$

Basic question:

How large can $\widehat{f}$ be?

The Fourier Restriction Problem

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} d\xi.$$

Better question:

Let $1 \leqslant p, q \leqslant \infty$. When do we have $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$?

The Fourier Restriction Problem

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} d\xi.$$

Better question:

Let $1 \leqslant p, q \leqslant \infty$. When do we have $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$?

- If $p = 2$, then Plancherel gives $\|\widehat{f}\|_{L^2} \approx \|f\|_{L^2}$.

The Fourier Restriction Problem

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} d\xi.$$

Better question:

Let $1 \leqslant p, q \leqslant \infty$. When do we have $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$?

- If $p = 2$, then Plancherel gives $\|\widehat{f}\|_{L^2} \approx \|f\|_{L^2}$.
- If $p = \infty$, then $\|\widehat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$.

The Fourier Restriction Problem

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} d\xi.$$

Better question:

Let $1 \leqslant p, q \leqslant \infty$. When do we have $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$?

- If $p = 2$, then Plancherel gives $\|\widehat{f}\|_{L^2} \approx \|f\|_{L^2}$.
- If $p = \infty$, then $\|\widehat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$.
- Interpolating then gives

$$\frac{1}{q} + \frac{1}{p} = 1 \text{ and } 1 \leqslant p \leqslant 2, \qquad \Longrightarrow \qquad \|\widehat{f}\|_{L^q} \lesssim \|f\|_{L^p}.$$

In fact this is only possibility (Hausdorf-Young Inequality), so the story is complete.

The Fourier Restriction Problem

What if only want to estimate the size of $\widehat{f}$ on a “small” set?

Suppose $S \subset \mathbb{R}^n$ is a hypersurface. For which $1 \leqslant p, q \leqslant \infty$ do we have

$$\|\widehat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}?$$

The Fourier Restriction Problem

What if only want to estimate the size of $\widehat{f}$ on a "small" set?

Suppose $S \subset \mathbb{R}^n$ is a hypersurface. For which $1 \leqslant p, q \leqslant \infty$ do we have

$$\|\widehat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}?$$

This type of question arises in PDE, number theory, geometric measure theory ...

- $p = 2$ not possible since $\widehat{f}$ only an $L^2(\mathbb{R}^n)$ function, can't restrict to set of measure zero!
- $p = 1$, then $\widehat{f}$ continuous $\Longrightarrow$ restriction $\widehat{f}|_S$ is well-defined and belongs to $L^\infty(\mathbb{S}^{n-1})$.

The Fourier Restriction Problem

What if only want to estimate the size of $\widehat{f}$ on a "small" set?

Suppose $S \subset \mathbb{R}^n$ is a hypersurface. For which $1 \leqslant p, q \leqslant \infty$ do we have

$$\|\widehat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}?$$

What about $1 < p < 2$?

- Restriction to the plane not possible!

  Suppose $S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and take

  $$f(x) = \psi(x_1, \dots, x_{n-1}, \lambda^{-1} x_n)$$

  with $\psi \in C_0^\infty$ and $\psi \neq 0$.

  Then

  $$\|f\|_{L^p(\mathbb{R}^n)} \approx \lambda^{\frac{1}{p}}, \qquad \|\widehat{f}\|_{L^q(S)} \gtrsim \lambda$$

  hence if restriction bound holds $\lambda \lesssim \lambda^{\frac{1}{p}}$. Letting $\lambda \to \infty$ gives $p = 1$.

The Fourier Restriction Problem

What if only want to estimate the size of $\widehat{f}$ on a "small" set?

Suppose $S \subset \mathbb{R}^n$ is a hypersurface. For which $1 \leqslant p, q \leqslant \infty$ do we have

$$\|\widehat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}?$$

What about $1 < p < 2$?

- Restriction to the plane not possible!

  Suppose $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and take

  $$f(x) = \psi(x_1, \ldots, x_{n-1}, \lambda^{-1} x_n)$$

  with $\psi \in C_0^\infty$ and $\psi \neq 0$.

  Then

  $$\|f\|_{L^p(\mathbb{R}^n)} \approx \lambda^{\frac{1}{p}}, \qquad \|\widehat{f}\|_{L^q(S)} \gtrsim \lambda$$

  hence if restriction bound holds $\lambda \lesssim \lambda^{\frac{1}{p}}$. Letting $\lambda \to \infty$ gives $p = 1$.

Stein 60's: true for some $1 < p < 2$ in the case of the sphere $S = \mathbb{S}^{n-1}$.

The Extension Problem

Write surface $S = \{(\Phi(\xi), \xi) \mid \xi \in \mathbb{R}^d\}$ (so $d = n-1$). Given $f : \mathbb{R}^d \to \mathbb{C}$ define extension operator

$$(\mathcal{E}_S f)(t,x) = \int_{\mathbb{R}^d} e^{i(t\Phi(\xi) + x\cdot\xi)} f(\xi) d\xi.$$

Fourier Restriction Conjecture - Stein 60's

Assume S is (compact) surface with non-vanishing Gaussian curvature. If $\frac{1}{q} < \frac{d}{2(d+1)}$ and $\frac{1}{q} \leqslant \frac{d}{d+2}(1 - \frac{1}{p})$ we have

$$\|\mathcal{E}_S f\|_{L^q_{t,x}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

The Extension Problem

Write surface $S = \{(\Phi(\xi), \xi) \mid \xi \in \mathbb{R}^d\}$ (so $d = n - 1$). Given $f : \mathbb{R}^d \to \mathbb{C}$ define extension operator

$$(\mathcal{E}_S f)(t, x) = \int_{\mathbb{R}^d} e^{i(t\Phi(\xi) + x\cdot\xi)} f(\xi) d\xi.$$

Fourier Restriction Conjecture - Stein 60's

Assume S is (compact) surface with non-vanishing Gaussian curvature. If $\frac{1}{q} < \frac{d}{2(d+1)}$ and $\frac{1}{q} \leqslant \frac{d}{d+2}(1 - \frac{1}{p})$ we have

$$\|\mathcal{E}_S f\|_{L^q_{t,x}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

- Stein-Thomas estimate: if $S = \mathbb{S}^d$ and $\frac{1}{q} \leqslant \frac{d}{2(d+2)}$ then we have

$$\|\mathcal{E}_S f\|_{L^q_{t,x}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

- Take $d = 4$, this gives

$$\|\mathcal{E}_S f\|_{L^3_{t,x}(\mathbb{R}^{1+4})} \lesssim \|f\|_{L^2(\mathbb{R}^4)}.$$

The Extension Problem

Write surface $S = \{(\Phi(\xi), \xi) \mid \xi \in \mathbb{R}^d\}$ (so $d = n - 1$). Given $f : \mathbb{R}^d \to \mathbb{C}$ define extension operator

$$(\mathcal{E}_S f)(t, x) = \int_{\mathbb{R}^d} e^{i(t\Phi(\xi) + x\cdot\xi)} f(\xi) d\xi.$$

Fourier Restriction Conjecture - Stein 60's

Assume S is (compact) surface with non-vanishing Gaussian curvature. If $\frac{1}{q} < \frac{d}{2(d+1)}$ and $\frac{1}{q} \leqslant \frac{d}{d+2}(1 - \frac{1}{p})$ we have

$$\|\mathcal{E}_S f\|_{L^q_{t,x}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

- If $\Phi(\xi) = |\xi|^2$, then S is a paraboloid, and $u(t, x) = \mathcal{E}_S f(t, x)$ gives solution to Schrödinger equation

$$i\partial u + \Delta u = 0.$$

  In particular, Stein-Thomas estimate gives the Strichartz estimate

$$\|u\|_{L^3_{t,x}(\mathbb{R}^{1+4})} \lesssim \|f\|_{L^2(\mathbb{R}^4)}.$$

Why does curvature help?

Fix $\delta \ll 1$ and suppose f supported in ball $B_\delta(\xi_0) = \{|\xi - \xi_0| \leqslant \delta\}$. Note that for $\xi \in B_\delta(\xi_0)$ we have Taylor series expansion

$$\Phi(\xi) = \Phi(\xi_0) + \nabla\Phi(\xi_0) \cdot (\xi - \xi_0) + \mathcal{O}(\delta^2).$$

Why does curvature help?

Fix $\delta \ll 1$ and suppose f supported in ball $B_\delta(\xi_0) = \{|\xi - \xi_0| \leqslant \delta\}$. Note that for $\xi \in B_\delta(\xi_0)$ we have Taylor series expansion

$$\Phi(\xi) = \Phi(\xi_0) + \nabla\Phi(\xi_0) \cdot (\xi - \xi_0) + \mathcal{O}(\delta^2).$$

In particular, provided $|t| \ll \delta^{-2}$ short computation gives

$$\begin{aligned} |(\mathcal{E}_S f)(t,x)| = \Big| \int_{\mathbb{R}^d} e^{i(t\Phi(\xi) + x\cdot\xi)} f(\xi) d\xi \Big| &\approx \Big| \int_{\mathbb{R}^d} e^{i(x + t\nabla\Phi(\xi_0))\cdot\xi} f(\xi) d\xi \Big| \\ &= |\widehat{f}(x + t\nabla\Phi(\xi_0))|. \end{aligned}$$

Why does curvature help?

Fix $\delta \ll 1$ and suppose f supported in ball $B_\delta(\xi_0) = \{|\xi - \xi_0| \leqslant \delta\}$. Note that for $\xi \in B_\delta(\xi_0)$ we have Taylor series expansion

$$\Phi(\xi) = \Phi(\xi_0) + \nabla\Phi(\xi_0) \cdot (\xi - \xi_0) + \mathcal{O}(\delta^2).$$

In particular, provided $|t| \ll \delta^{-2}$ short computation gives

$$\begin{aligned} |(\mathcal{E}_S f)(t,x)| = \Big| \int_{\mathbb{R}^d} e^{i(t\Phi(\xi)+x\cdot\xi)} f(\xi) d\xi \Big| &\approx \Big| \int_{\mathbb{R}^d} e^{i(x+t\nabla\Phi(\xi_0))\cdot\xi} f(\xi) d\xi \Big| \\ &= |\widehat{f}(x + t\nabla\Phi(\xi_0))|. \end{aligned}$$

In other words:

If $\operatorname{supp} f \subset B_\delta(\xi_0)$ then for times $|t| \ll \delta^{-2}$, extension operator just translates Fourier transform $\widehat{f}$ by $t\nabla\Phi(\xi_0)$

Key observation: if we also have $\operatorname{supp} \widehat{f} \subset B_{\delta^{-1}}(0)$ say, then at least for times $|t| \leqslant \delta^{-2}$ we have $\operatorname{supp} \mathcal{E}_S f$ contained in 'tube'

$$\{(t,x) \in \mathbb{R} \times \mathbb{R}^d \mid |x + t\nabla\Phi(\xi_0)| \leqslant \delta^{-1}\}.$$

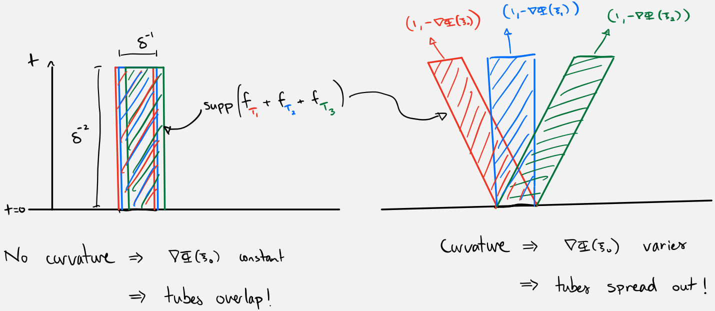
Why does curvature help?

Decomposing general f gives wave packet decomposition: for times $|t| \leqslant \delta^{-2}$ we have

$$\mathcal{E}_S f(t,x) = \sum_{T \in \mathcal{T}} a_T f_T, \qquad f_T(t,x) \approx \widehat{f_{x_T,\xi_T}}(x + t\nabla\Phi(\xi_T))$$

where

- $\mathcal{T}$ is collection of tubes $T \subset \mathbb{R}^{1+d}$ size $\delta^{-2} \times \delta^{-1}$ oriented in direction $(1, -\nabla\Phi(\xi_0))$
- Coefficients $|a_T| \approx 1$ and $\operatorname{supp} f_T \subset T$



Beyond the Stein-Thomas exponent

Fourier Restriction Conjecture - Stein 60's

Assume S is compact surface with non-vanishing Gaussian curvature. $\frac{1}{q} < \frac{d}{2(d+1)}$ and $\frac{1}{q} \leqslant \frac{d}{d+2}(1 - \frac{1}{p})$ we have

$$\|\mathcal{E}_S f\|_{L^q_{t,x}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

- Known in the case $d = 1$: Fefferman-Stein'70, Zygmund'74,

Fix $d = 2$. Conjectured range is then $q > 3$ (setting $\frac{1}{q} = \frac{1}{2}(1 - \frac{1}{p})$)

- Stein-Thomas range: $q \geqslant 4$
- "Bilinear implies linear": $3 + \frac{1}{3} < q \leqslant 4$
  Bourgain'91, Tao-Vargas'00, Wolff'01, Tao'01, ...
- "Multilinear implies linear": $3 + \frac{1}{7} < q \leqslant 3 + \frac{1}{3}$
  Bourgain-Guth'11, Guth'16, Wang-Wu'24, ...
- Remains open despite much recent progress...

Bilinear Restriction Estimates

Define

$$e^{it|\nabla|}f = \int_{\mathbb{R}^n} e^{i(t|\xi|+x\cdot\xi)}\widehat{f}(\xi)d\xi$$

(this is essentially the extension operator for the cone).

Theorem (Tao'01)

Let $\frac{n+3}{n+1} \leqslant q \leqslant 2$, $\lambda > 1$ *and* $\epsilon > 0$. *If*

$$\operatorname{supp}\widehat{f} \subset \{|\xi - e_1| \ll 1\}, \qquad \operatorname{supp}\widehat{g} \subset \{|\xi - \lambda e_2| \ll \lambda\}$$

then

$$\|e^{it|\nabla|}f e^{it|\nabla|}g\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{q}-\frac{1}{2}+\epsilon}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

- The Fourier support assumption implies that the surfaces are transverse.
- The first bilinear restriction estimate using both transversality and curvature due to Bourgain'91, full non-endpoint range is due to Wolff'01.

Bilinear Restriction Estimates

Theorem (Tao'01)

Let $\frac{n+3}{n+1} \leqslant q \leqslant 2$, $\lambda > 1$ *and* $\epsilon > 0$. *If*

$$\operatorname{supp} \widehat{f} \subset \{|\xi - e_1| \ll 1\}, \qquad \operatorname{supp} \widehat{g} \subset \{|\xi - \lambda e_2| \ll \lambda\}$$

then

$$\|e^{it|\nabla|} f e^{it|\nabla|} g\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{q} - \frac{1}{2} + \epsilon} \|f\|_{L^2} \|g\|_{L^2}.$$

Why is transversality helpful?

surfaces transverse $\implies$ waves propagate in transverse directions

which means better decay. For instance, on $\mathbb{R}^{1+1}$ we have

$$\|f(x+t)g(x-t)\|_{L^q_{t,x}(\mathbb{R}^{1+1})} \approx \|f\|_{L^q(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}$$

but

$$\|f(x+t)g(x+t)\|_{L^q_{t,x}(\mathbb{R}^{1+1})} = \infty.$$

Bilinear Restriction Estimates

Theorem (Tao'01)

Let $\frac{n+3}{n+1} \leqslant q \leqslant 2$*,* $\lambda > 1$ *and* $\epsilon > 0$*. If*

$$\operatorname{supp} \widehat{f} \subset \{|\xi - e_1| \ll 1\}, \qquad \operatorname{supp} \widehat{g} \subset \{|\xi - \lambda e_2| \ll \lambda\}$$

then

$$\|e^{it|\nabla|} f e^{it|\nabla|} g\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{q} - \frac{1}{2} + \epsilon} \|f\|_{L^2} \|g\|_{L^2}.$$

- Originally, bilinear restriction estimates used to make progress on linear restriction problem.
- Would like to apply result to nonlinear PDE. Two problems:

Bilinear Restriction Estimates

Theorem (Tao'01)

Let $\frac{n+3}{n+1} \leqslant q \leqslant 2$, $\lambda > 1$ *and* $\epsilon > 0$. *If*

$$\operatorname{supp} \widehat{f} \subset \{|\xi - e_1| \ll 1\}, \qquad \operatorname{supp} \widehat{g} \subset \{|\xi - \lambda e_2| \ll \lambda\}$$

then

$$\|e^{it|\nabla|} f e^{it|\nabla|} g\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{q} - \frac{1}{2} + \epsilon} \|f\|_{L^2} \|g\|_{L^2}.$$

- Originally, bilinear restriction estimates used to make progress on linear restriction problem.
- Would like to apply result to nonlinear PDE. Two problems:
    - derivative loss

Bilinear Restriction Estimates

Theorem (Tao'01)

Let $\frac{n+3}{n+1} \leqslant q \leqslant 2$, $\lambda > 1$ *and* $\epsilon > 0$. *If*

$$\operatorname{supp} \widehat{f} \subset \{|\xi - e_1| \ll 1\}, \qquad \operatorname{supp} \widehat{g} \subset \{|\xi - \lambda e_2| \ll \lambda\}$$

then

$$\|e^{it|\nabla|} f e^{it|\nabla|} g\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{q} - \frac{1}{2} + \epsilon} \|f\|_{L^2} \|g\|_{L^2}.$$

- Originally, bilinear restriction estimates used to make progress on linear restriction problem.
- Would like to apply result to nonlinear PDE. Two problems:
  - derivative loss
  - only holds for free waves

Bilinear Restriction Estimates

Theorem (C.'19)

Let $\frac{n+3}{n+1} < q \leqslant 2$, $\lambda > 1$. *If* $\operatorname{supp} \widehat{f} \subset \{|\xi - e_1| \ll 1\}$, $\operatorname{supp} \widehat{g} \subset \{|\xi - \lambda e_2| \ll \lambda\}$ *then*

$$\|e^{it|\nabla|} f e^{it|\nabla|} g\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{q} - \frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}.$$

- No derivative loss as long as $q > \frac{n+3}{n+1}$.
- In fact C.'19 contains a much more general result, which gives a sharp bilinear restriction estimate for general phase functions (including solutions to Schrödinger equation).

This solves first problem!

Atomic Bilinear Restriction Estimates

Theorem (C.'19)

Let $\frac{n+3}{n+1} < q \leqslant 2$, $\lambda > 1$, *and* $2 \leqslant b < \frac{2}{(n+1)q}$. *If* $\operatorname{supp}\widehat{u} \subset \{|\xi - e_1| \ll 1\}$, $\operatorname{supp}\widehat{v} \subset \{|\xi - \lambda e_2| \ll \lambda\}$ *then*

$$\|uv\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{p}-\frac{1}{2}} \|u\|_{U^2_{|\nabla|}} \|v\|_{U^b_{|\nabla|}}.$$

- The atomic space $U^b_{|\nabla|}$ is a Banach space of right-continuous functions $u : \mathbb{R}^{1+n} \to \mathbb{C}$ which are "close" to free solutions to the wave equation.
- Typical element $u \in U^b$ is

$$u(t,x) = \sum_{j=1}^{N} \mathbb{1}_{[t_j, t_{j+1})}(t) e^{it|\nabla|} f_j, \qquad t_1 < t_2 < \cdots < t_N < t_{N+1} = \infty$$

where $\left(\sum_{j=1}^N \|f_j\|_{L^2}^b \right)^{\frac{1}{b}} < \infty$. In particular $e^{it|\nabla|} f \in U^b$.

This solves second problem!

Atomic Bilinear Restriction Estimates

Theorem (C.'19)

Let $\frac{n+3}{n+1} < q \leqslant 2$, $\lambda > 1$, *and* $2 \leqslant b < \frac{2}{(n+1)q}$. *If* $\operatorname{supp} \widehat{u} \subset \{|\xi - e_1| \ll 1\}$, $\operatorname{supp} \widehat{v} \subset \{|\xi - \lambda e_2| \ll \lambda\}$ *then*

$$\|uv\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \lesssim \lambda^{\frac{1}{p} - \frac{1}{2}} \|u\|_{U^2_{|\nabla|}} \|v\|_{U^b_{|\nabla|}}.$$

Summary:

- Removes derivative loss from result of Tao (at least away from endpoint).
- Shows that bilinear restriction estimates hold for functions "close" to free solutions.
  - Extends earlier result of C.-Herr'16 in case $\lambda = 1$.
- Applications
  - Dispersive PDE:
    C.-Herr'18, Shen-Soffer-Wu'22, C.-Herr-Nakanishi'22.
  - Restriction problem for surfaces with degenerate curvature:
    Carneiro-Sousa-Stovall'18, Buschenhenke-Müller-Vargas'21.

The Zakharov system

$$i\partial_t u + \Delta u = vu,$$
$$\frac{1}{\alpha^2}\partial_t^2 v - \Delta v = \Delta |u|^2$$

where $\alpha \in \mathbb{R}$, $u : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$ and $v : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$.

- Derived as a mathematical model for Langmuir waves in plasma physics by Zakharov1972. Here u denotes the envelope of an electric field, and v is the ion density.
- Taking the subsonic limit $\alpha \to \infty$, formally gives the (focusing) cubic nonlinear Schrödinger equation

  $$i\partial_t u + \Delta u = -|u|^2 u.$$

  In certain regimes this convergence has been demonstrated rigourously Schochet-Weinstein1986, Masmoudi-Nakanishi2008.

Hamiltonian formulation

$$i\partial_t u + \Delta u = \Re(V)u,$$
$$i\partial_t V - |\nabla|V = |\nabla||u|^2$$

where $(u, V) : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$ and $|\nabla| = \sqrt{-\Delta}$ (define via say Fourier transform).

The Hamiltonian (or energy) and mass are

$$\mathcal{E}_Z(u, V) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|V|^2 + \frac{1}{2}\Re(V)|u|^2 dx, \qquad \mathcal{M}(u) = \int_{\mathbb{R}^d} \frac{1}{2}|u|^2 dx.$$

Both energy and mass conserved under the nonlinear flow

$$\mathcal{E}_Z\big(u(t), v(t)\big) = \mathcal{E}_Z\big(u(0), V(0)\big)$$

and

$$\mathcal{M}\big(u(t)\big) = \mathcal{M}\big(u(0)\big).$$

Cauchy problem

$$i\partial_t u + \Delta u = \Re(V)u,$$
$$i\partial_t V - |\nabla|V = |\nabla||u|^2,$$
$$(u, V)(0) = (f, g).$$

Fix $d = 4$ and assume data (f, g) has finite energy and mass

$$\mathcal{E}_Z(f, g) + \mathcal{M}(f) < \infty.$$

Questions:

- Can we prove solution (u, V) exists globally in time?
- Do the linear dynamics dominate as $t \to \pm\infty$?

Cauchy problem

$$i\partial_t u + \Delta u = \Re(V)u,$$
$$i\partial_t V - |\nabla|V = |\nabla||u|^2,$$
$$(u, V)(0) = (f, g).$$

Fix $d = 4$ and assume data (f, g) has finite energy and mass

$$\mathcal{E}_Z(f, g) + \mathcal{M}(f) < \infty.$$

Goal: If data has finite energy and mass, then solutions are global and scatter as $t \to \infty$.

- Scattering means that linear terms dominate as $t \to \infty$.
  In other words, the solution (u, V) converges to a solution (u_∞, V_∞) to the linear problem

$$i\partial_t u_\infty + \Delta u_\infty = 0,$$
$$i\partial_t V_\infty - |\nabla|V_\infty = 0.$$

Obstructions I: Blow-up

$$i\partial_t u + \Delta u = \Re(V)u,$$
$$i\partial_t V - |\nabla|V = |\nabla||u|^2,$$
$$(u, V)(0) = (f, g).$$

Analogous to the NLS case, solutions can blowup in finite time

- Merle1996 If $d = 2, 3$ then all radial solutions with negative energy $E_Z(u, V) < 0$ blow-up (either in finite time, or infinite time).
- Holmer2007 Ill-posedness for "unbalanced" data when $d = 1$.
- Guo-Nakanishi2021 In $d = 4$ (weak) blowup/nonscattering for large energy.
- Krieger-Schmid2024 Construction of blowup solutions in a neighbourhood of 'ground state'
$$Q(x) = \frac{8}{8 + |x|^2}.$$

Obstructions II: Solitary Waves/Solitons

$$\begin{aligned} i\partial_t u + \Delta u &= \Re(V)u, \\ i\partial_t V - |\nabla| V &= |\nabla| |u|^2, \\ (u, V)(0) &= (f, g). \end{aligned}$$

Again, as in the NLS case:

- Let $Q(x) = \frac{8}{8+|x|^2}$. Then $\Delta Q = -Q^3$

  $$(u, V)(t) = (Q, -Q^2)$$

  gives a stationary solution to the Zakharov equation.
- Same obstruction as in the NLS case, and the ground state Q plays an identical role here (smallest energy non-dispersive solution).

Obstructions II: Solitary Waves/Solitons

$$i\partial_t u + \Delta u = \Re(V)u,$$
$$i\partial_t V - |\nabla|V = |\nabla||u|^2,$$
$$(u, V)(0) = (f, g).$$

In view of what we know about the NLS, the following conjecture seems reasonable.

Conjecture: global existence and scattering below ground state

If the data (f, g) has energy and mass below the ground state Q, then solution (u, V) is global and scatters to a linear solution as $t \to \infty$.

- Conjecture essentially states that the 'soliton' Q is smallest energy non-dispersive solution.
- Unlike the corresponding question for the NLS equation, conjecture is still unresolved.

Global Well-posedness Below the Ground State: Radial Case

Theorem (Guo-Nakanishi'21)

If data (f, g) is radial and below the ground state

$$\mathcal{E}_Z(f, g) < \mathcal{E}_Z(Q, -Q^2), \qquad \|g\|_{L^2} < \|Q^2\|_{L^2}$$

then solution (u, V) is global and scatters.

- Lots of previous work!
  Bourgain-Colliander'96, Ginibre-Tsutsumi-Velo'97, Colliander-Holmer-Tzirakis'08, Bejenaru-Herr-Holmer-Tataru'08, Bejenaru-Herr'10, ...
- Energy is coercive below the ground state, in fact we have

$$\mathcal{E}_Z(f, g) \approx \|\nabla f\|_{L^2}^2 + \|g\|_{L^2}^2.$$

This is not true without the ground state constraint.

Sebastian Herr (Bielefeld) and Kenji Nakanishi (RIMS - Kyoto).

Non-radial data

Theorem (C.-Herr-Nakanishi'21, C.-Herr-Nakanishi'23)

If data (f, g) is below the ground state

$$\mathcal{E}_Z(f, g) < \mathcal{E}_Z(Q, -Q^2), \qquad \|g\|_{L^2} < \|Q^2\|_{L^2}$$

then solution (u, V) exists globally in time.

Non-radial data

Theorem (C.-Herr-Nakanishi'21, C.-Herr-Nakanishi'23)

If data (f, g) is below the ground state

$$\mathcal{E}_Z(f, g) < \mathcal{E}_Z(Q, -Q^2), \qquad \|g\|_{L^2} < \|Q^2\|_{L^2}$$

then solution (u, V) exists globally in time.

- Only shows that below the ground state Q, solutions cannot form singularities in finite time.
- Does not give any information on the dynamics.

  For instance, we cannot rule out the possibility of 'exotic' non-dispersive solutions below the ground state...

Partial Progress

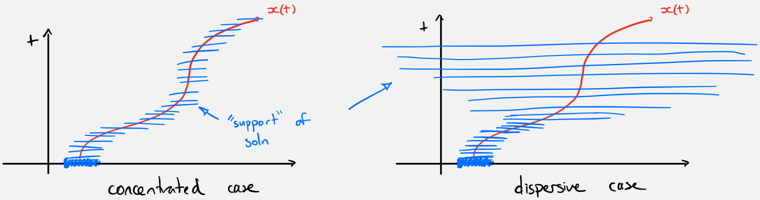
Theorem (C.2024)

If scattering fails, then there must exist a global solution (ψ, ϕ) to the Zakharov equation and a trajectory $x(t) : \mathbb{R} \to \mathbb{R}^4$ such that

(1) The solution (ψ, ϕ) lies below the ground state

$$\mathcal{E}_Z(\psi, \phi) < \mathcal{E}(Q, -Q^2), \qquad \sup_{t\in\mathbb{R}} \|\phi(t)\|_{L^2(\mathbb{R}^4)} < \|Q^2\|_{L^2(\mathbb{R}^4)}.$$

(2) The solutions $(\psi, \phi)(t, x)$ are stationary modulo translations, in the sense that they remain concentrated in some neighborhood of the trajectory $x(t)$.



Partial Progress

Theorem (C.2024)

If scattering fails, then there must exist a global solution (ψ, ϕ) to the Zakharov equation and a trajectory $x(t) : \mathbb{R} \to \mathbb{R}^4$ such that

(1) The solution (ψ, ϕ) lies below the ground state

$$\mathcal{E}_Z(\psi, \phi) < \mathcal{E}(Q, -Q^2), \qquad \sup_{t \in \mathbb{R}} \|\phi(t)\|_{L^2(\mathbb{R}^4)} < \|Q^2\|_{L^2(\mathbb{R}^4)}.$$

(2) The solutions $(\psi, \phi)(t, x)$ are stationary modulo translations, in the sense that they remain concentrated in some neighborhood of the trajectory $x(t)$.

- The solution (ψ, ϕ) essentially behaves like a translated version of the ground state solution $(Q, -Q^2)(x - x(t))$.
- Theorem reduces conjecture to proving that these solution cannot exist!

  This is reasonable, variational arguments show that Q is the 'smallest' solution to $\Delta Q = -Q^3$.

Partial Progress

Theorem (C.2024)

If scattering fails, then there must exist a global solution (ψ, ϕ) to the Zakharov equation and a trajectory $x(t) : \mathbb{R} \to \mathbb{R}^4$ such that

(1) The solution (ψ, ϕ) lies below the ground state

$$\mathcal{E}_Z(\psi, \phi) < \mathcal{E}(Q, -Q^2), \qquad \sup_{t \in \mathbb{R}} \|\phi(t)\|_{L^2(\mathbb{R}^4)} < \|Q^2\|_{L^2(\mathbb{R}^4)}.$$

(2) The solutions $(\psi, \phi)(t, x)$ are stationary modulo translations, in the sense that they remain concentrated in some neighborhood of the trajectory $x(t)$.

- In radial case, must have $x(t) = 0$, and can rule out solutions via a virial estimate which is (morally) of the form

$$\int_{\mathbb{R}} \int_{|x| \leqslant R} |\psi|^2 dx dt < \infty \qquad \texttt{Guo-Nakanishi2022}.$$

Partial Progress

Theorem (C.2024)

If scattering fails, then there must exist a global solution (ψ, ϕ) to the Zakharov equation and a trajectory $x(t) : \mathbb{R} \to \mathbb{R}^4$ such that

(1) The solution (ψ, ϕ) lies below the ground state

$$\mathcal{E}_Z(\psi, \phi) < \mathcal{E}(Q, -Q^2), \qquad \sup_{t\in\mathbb{R}} \|\phi(t)\|_{L^2(\mathbb{R}^4)} < \|Q^2\|_{L^2(\mathbb{R}^4)}.$$

(2) The solutions $(\psi, \phi)(t, x)$ are stationary modulo translations, in the sense that they remain concentrated in some neighborhood of the trajectory $x(t)$.

- In radial case, must have $x(t) = 0$, and can rule out solutions via a virial estimate which is (morally) of the form

$$\int_{\mathbb{R}} \int_{|x|\leqslant R} |\psi|^2 dx dt < \infty \qquad \text{Guo-Nakanishi2022}.$$

Why? If solution stayed localised around $|x| \leqslant R$, then essentially have

$$\int_{|x|\leqslant R} |\psi|^2 dx \approx \text{constant} \quad \Rightarrow \quad \int_{\mathbb{R}} \int_{|x|\leqslant R} |\psi|^2 dx dt = \infty, \text{ contradiction!}$$

Ideas in proof: threshold

Simplify set up slightly:

Conjecture: If $\mathcal{E}_Z(u, V) < \mathcal{E}_Z(Q, -Q^2)$ then solution is global and scatters.

By small data theory, for small E, we have implication

$$\mathcal{E}_Z(f, g) < E \qquad \Longrightarrow \qquad \text{solution is global and scatters.}$$

Now consider what happens if we make E larger.

Ideas in proof: threshold

Simplify set up slightly:

Conjecture: If $\mathcal{E}_Z(u, V) < \mathcal{E}_Z(Q, -Q^2)$ then solution is global and scatters.

By small data theory, for small E, we have implication

$$\mathcal{E}_Z(f, g) < E \qquad \Longrightarrow \qquad \text{solution is global and scatters.}$$

Now consider what happens if we make E larger.

Can't take $E = \mathcal{E}_Z(Q, -Q^2)$ as we assume conjecture fails!

Ideas in proof: threshold

Simplify set up slightly:

Conjecture: If $\mathcal{E}_Z(u, V) < \mathcal{E}_Z(Q, -Q^2)$ then solution is global and scatters.

By small data theory, for small E, we have implication

$$\mathcal{E}_Z(f, g) < E \qquad \Longrightarrow \qquad \text{solution is global and scatters.}$$

Now consider what happens if we make E larger.

Can't take $E = \mathcal{E}_Z(Q, -Q^2)$ as we assume conjecture fails!

So must exist some threshold $E^* < \mathcal{E}_Z(Q, -Q^2)$ such that

(1) If $\mathcal{E}_Z(f, g) < E^*$ then solution is global and scatters.

(2) Above threshold scattering fails.

Unpacking (2) we see that there exists sequence (f_n, g_n) s.t. $\lim_{n\to\infty} \mathcal{E}_Z(f_n, g_n) = E^*$ and corresponding solution (u_n, V_n) does not scatter.

Ideas in proof: concentration compactness

(1) If $\mathcal{E}_Z(f,g) < E^*$ then solution is global and scatters.

(2) There exists sequence (f_n, g_n) s.t. $\lim_{n\to\infty} \mathcal{E}_Z(f_n, g_n) = E^*$ and solution (u_n, V_n) does not scatter.

Next step is to extract convergent subsequence from (f_n, g_n).

Ideas in proof: concentration compactness

(1) If $\mathcal{E}_Z(f,g) < E^*$ then solution is global and scatters.
(2) There exists sequence (f_n, g_n) s.t. $\lim_{n\to\infty} \mathcal{E}_Z(f_n, g_n) = E^*$ and solution (u_n, V_n) does not scatter.

Next step is to extract convergent subsequence from (f_n, g_n).

- Sequence is bounded and bounded sequences in finite dimensional spaces have convergent subsequences (Bolzano-Weierstrass!)

Ideas in proof: concentration compactness

(1) If $\mathcal{E}_Z(f, g) < E^*$ then solution is global and scatters.
(2) There exists sequence (f_n, g_n) s.t. $\lim_{n\to\infty} \mathcal{E}_Z(f_n, g_n) = E^*$ and solution (u_n, V_n) does not scatter.

Next step is to extract convergent subsequence from (f_n, g_n).

- Sequence is bounded and bounded sequences in finite dimensional spaces have convergent subsequences (Bolzano-Weierstrass!)
- Unfortunately sequence is only bounded in infinite dimensional space

$$H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4).$$

- To extract limit (f, g) need to argue via concentration compactness which quantifies the loss of compactness (Lions1985, Gérard1998, Keraani2001).

Define (ψ, ϕ) as solution with data (f, g), then $\mathcal{E}_Z(\psi, \phi) = E^*$ lies on threshold.

Ideas in proof: solution at threshold must be concentrated

Final goal is to prove that (ψ, ϕ) is not dispersive in the sense that it remains concentrated around some trajectory $x(t)$.

How is this done? Well if (ψ, ϕ) was not concentrated, then we could decompose

$$(\psi, \phi) \approx (\psi_1, \phi_1) + (\psi_2, \phi_2)$$

with (ψ_j, ϕ_j) solutions concentrated in separated regions.

Ideas in proof: solution at threshold must be concentrated

Final goal is to prove that (ψ, ϕ) is not dispersive in the sense that it remains concentrated around some trajectory $x(t)$.

How is this done? Well if (ψ, ϕ) was not concentrated, then we could decompose

$$(\psi, \phi) \approx (\psi_1, \phi_1) + (\psi_2, \phi_2)$$

with (ψ_j, ϕ_j) solutions concentrated in separated regions.

But then

$$\mathcal{E}_Z(\psi, \phi) \approx \mathcal{E}_Z(\psi_1, \phi_1) + \mathcal{E}_Z(\psi_2, \phi_2)$$

and so (ψ_1, ϕ_1) and (ψ_2, ϕ_2) both have energy below threshold E^*

$$\Longrightarrow (\psi_j, \phi_j) \text{ scatter by definition of threshold } E^*$$

$$\Longrightarrow (\psi, \phi) \approx (\psi_1, \phi_1) + (\psi_2, \phi_2) \text{ also scatters.}$$

Contradiction!

Ideas in proof: solution at threshold must be concentrated

Final goal is to prove that (ψ, ϕ) is not dispersive in the sense that it remains concentrated around some trajectory $x(t)$.

- Above strategy closely related to original induction on energy argument introduced by Bourgain1999.
- Applying concentration compactness and extracting threshold solutions is a key tool in studying asymptotic behaviour of dispersive PDE Kenig-Merle2006, Killip-Visan2010, Dodson2019,...
- Implementation in Zakharov case difficult as required estimates are very delicate.

In fact progress on Zakharov equation only possible due to recent developments in Harmonic analysis which have lead to robust bilinear restriction estimates Tao2001, Lee-Vargas2008, Bejenaru2019, C.2019, ...

Quick summary of strategies

- Exploit iterative/perturbative arguments to prove implication

$$\text{Good estimates for linear problem} \implies \text{understand dynamics of nonlinear problem.}$$

Main issue: Only works when linear problem is a good approximation of nonlinear problem.

Quick summary of strategies

- Exploit iterative/perturbative arguments to prove implication

$$\text{Good estimates for linear problem} \implies \text{understand dynamics of nonlinear problem.}$$

Main issue: Only works when linear problem is a good approximation of nonlinear problem.

- To understand dynamics of large data we
  1. Run induction on energy to reduce to solutions at threshold

     $$\implies \text{solutions at threshold must concentrate.}$$

  2. Rule out concentrating/soliton like solutions via conservation laws/monotonicity formula.

Currently step (2) still work in progress for Zakarov equation.

References

- T. Candy, *Concentration compactness for the energy critical Zakharov system*, Discrete and Continuous Dynamical Systems **44** (2024), no. 5, (50 pages).
- T. Candy, S. Herr, and K. Nakanishi, *The Zakharov system in dimension* $d \geqslant 4$, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 8, (51 pages).
- T. Candy, S. Herr, and K. Nakanishi. *Global wellposedness for the energy-critical Zakharov system below the ground state*, Advances in Mathematics **384** (2021), (57 pages).
- T. Candy. *Multi-scale bilinear restriction estimates for general phases*, Math. Ann. **375** (2019), no. 1, (65 pages).

References

- T. Candy, *Concentration compactness for the energy critical Zakharov system*, Discrete and Continuous Dynamical Systems **44** (2024), no. 5, (50 pages).
- T. Candy, S. Herr, and K. Nakanishi, *The Zakharov system in dimension* $d \geqslant 4$, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 8, (51 pages).
- T. Candy, S. Herr, and K. Nakanishi. *Global wellposedness for the energy-critical Zakharov system below the ground state*, Advances in Mathematics **384** (2021), (57 pages).
- T. Candy. *Multi-scale bilinear restriction estimates for general phases*, Math. Ann. **375** (2019), no. 1, (65 pages).

Thank you for listening!